COHOMOLOGY OF LIE SUPERALGEBRAS $\mathfrak{sl}_{m|n}$ AND $\mathfrak{osp}_{2|2n}$

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Abstract. We explicitly compute the first and second cohomology groups of the classical Lie superalgebras $\mathfrak{sl}_{m|n}$ and $\mathfrak{osp}_{2|2n}$ with coefficients in the finite dimensional irreducible modules and the Kac modules. We also show that the second cohomology groups of these Lie superalgebras with coefficients in the respective universal enveloping algebras (under the adjoint action) vanish. The latter result in particular implies that the universal enveloping algebras $\mathrm{U}(\mathfrak{sl}_{m|n})$ and $\mathrm{U}(\mathfrak{osp}_{2|2n})$ do not admit any non-trivial formal deformations of Gerstenhaber type.

Contents

1. Introduction

We investigate the Lie superalgebra cohomology of the type I basic classical Lie superalgebras [7], namely, the special linear superalgebra $\mathfrak{sl}_{m|n}$, and the orthosymmplectic superalgebra $\mathfrak{osp}_{2|2n}$. Lie superalgebra cohomology was extensively studied by Fuks, Leites [3], and others (see [2] for a review). For any basic classical simple Lie superalgebra $\mathfrak{g}$, the cohomolgy groups $H^i(\mathfrak{g}, V)$ were computed [3, 2] for all i when the coefficient module V is $\mathbb{C}$ (even though relatively little seems to be known about these

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cohomolgy groups when the coefficient module is non-trivial). Variations of these cohomolgy groups, especially the relative cohomology groups and the cohomology groups of odd nilpotent subalgebras, have also been studied in depth because of their importance in the contexts of the Bott-Borel-Weil theory [13] and the Kazhdan-Lusztig theory for Lie supergroups [19, 1].

A motivation of this investigation and earlier work of one of us with Scheunert [16, 17] comes from the theory of quantum supergroups [11, 24, 25], the foundation of which lies in the deformation theory [4] of universal enveloping algebras of Lie superalgebras. Recall that the formal deformations of an associative algebra is classified by the second Hochschild cohomology group with coefficients in the algebra itself (regarded as a bi-module) [4]. In the case of the universal enveloping algebra of a Lie superalgebra, this Hochschild cohomology group can be shown to be isomorphic to the second Lie superalgebra cohomology group with coefficient module being the universal enveloping algebra under the adjoint action of the Lie superalgebra. Similarly, the first Lie superalgebra cohomology group with coefficients in the universal enveloping algebra controls the deformations of the co-algebra structure of the universal enveloping algebra.

One result of the present paper is Theorem 7.1 and (8.5) of Theorem 8.1, which states that for $\mathfrak{g}$ being $\mathfrak{sl}_{m|n}$ or $\mathfrak{osp}_{2|2n}$, $H^1(\mathfrak{g}, \mathrm{U}(\mathfrak{g})) \neq 0$, but $H^2(\mathfrak{g}, \mathrm{U}(\mathfrak{g})) = 0$. The vanishing of the second cohomology group implies that $\mathrm{U}(\mathfrak{sl}_{m|n})$ and $\mathrm{U}(\mathfrak{osp}_{2|2n})$ are rigid in the sense of [4]. Therefore, the Drinfeld versions of the quantized universal enveloping algebras of $\mathfrak{sl}_{m|n}$ and $\mathfrak{osp}_{2|2n}$ defined with any choice of Borel subalgebras are isomorphic to the universal enveloping algebras themselves over the power series ring. (This was proved for the special case of $\mathfrak{sl}_{m|1}$ in [17].) Also $H^1(\mathfrak{g}, \mathrm{U}(\mathfrak{g})) \neq 0$ implies that the co-algebra structure of $\mathrm{U}(\mathfrak{g})$ admits non-trivial deformations, a fact which is known from specific examples.

Another main result of this paper is the computation of the first and second Lie superalgebra cohomology groups of $\mathfrak{sl}_{m|n}$ and $\mathfrak{osp}_{2|2n}$ with coefficients in the finite dimensional Kac modules and the finite dimensional irreducible modules, which is summarized in Theorems 3.1, 4.1, 5.4, 6.18, and 8.1. This result is of intrinsic interest to the understanding of extensions of modules of these Lie superalgebras, and also extensions of the Lie superalgebras themselves. As a matter of fact, in his foundational paper [7] on the theory of Lie superalgebras, Kac posed the problem of determining the first cohomology groups of basic classical simple Lie superalgebras with coefficients in the finite dimensional irreducible modules (see also [8, 9]). Part of the paper solves the problem for the Lie superalgebras $\mathfrak{sl}_{m|n}$ and $\mathfrak{osp}_{2|2n}$. As we have alluded to earlier, when the coefficient module V is not $\mathbb{C}$, little seems to be known about the cohomolgy groups $H^i(\mathfrak{g}, V)$ for the basic classical Lie superalgebras; the main results (Theorems 3.1, 4.1, 5.4, 6.18, 7.1, and 8.1) of the present paper appear to be new.

The computations of the cohomology groups are carried out in this paper at an elementary level by exploring long exact sequences of cohomology groups arising from short exact sequences of modules, and also by using some elements of the Hochschild spectral sequence associated with the maximal even subalgebras of the Lie superalgebras. The computations also rely heavily on detailed knowledge on structures of Kac modules. We may mention that the analysis of structures of Kac modules is a technical and difficult problem. This renders Subsection 6.2 rather technical.

The organization of the paper is as follows. Sections 2 to 7 treat the Lie superalgebra cohomology groups of the special linear superalgebra in detail. Section 2 provides some necessary background material on $\mathfrak{sl}_{m|n}$. Sections 3 and 4 respectively present

the computations of the first and second cohomology groups of $\mathfrak{sl}_{m|n}$ with coefficients in finite dimensional Kac modules. Section 5 is devoted to the computation of the first cohomology groups of $\mathfrak{sl}_{m|n}$ with coefficients in finite dimensional irreducible modules, where we make use of the concepts of atypicality matrices, northeast chains (NE) of a weight, and n-, q-, c-relationships of atypical roots, which are all explained in Appendix A. In Section 6 we calculate the second cohomology groups of $\mathfrak{sl}_{m|n}$ with coefficients in finite dimensional irreducible modules. This section is divided into three subsections. Subsection 6.1 introduces the notion of primitive weight graphs, which is very useful for studying the structure of indecomposable $\mathfrak{sl}_{m|n}$-modules such as Kac modules. In Subsection 6.2 we analyse structures of some Kac modules of $\mathfrak{sl}_{m|n}$ and establish a series of technical lemmas needed for proving the main result on the second cohomology groups of $\mathfrak{sl}_{m|n}$ with coefficients in finite dimensional irreducible modules. Subsection 6.3 proves the main result of Section 6. Section 7 treats the second cohomology groups of $\mathfrak{sl}_{m|n}$ with coefficients in the universal enveloping algebra. Finally, in Section 8 we present the results on the cohomology groups of $\mathfrak{osp}_{2|2n}$, while omitting most of the technical details.

2. Preliminaries on the special linear superalgebra

2.1. **The special linear superalgebra.** We present some background material on the special linear superalgebra here and refer to [7, 14, 10] for more details. For general notions of graded vector spaces and graded algebraic structures we refer to the classic paper [12] by Milnor and Moore.

We shall work over the complex number field $\mathbb{C}$ throughout the paper. Given a $\mathbb{Z}_2$-graded vector space $W = W_{\bar 0} \oplus W_{\bar 1}$, we call $W_{\bar 0}$ and $W_{\bar 1}$ the even and odd subspaces, respectively. The elements of $W_{\bar 0} \cup W_{\bar 1}$ will be called homogeneous. Define a map $[\ ] : W_{\bar 0} \cup W_{\bar 1} \to \mathbb{Z}_2$ by $[w] = \bar i$ if $w \in W_{\bar i}$. For any two $\mathbb{Z}_2$-graded vector spaces V and W, the space of morphisms $\mathrm{Hom}_{\mathbb{C}}(V,W)$ is also $\mathbb{Z}_2$-graded with $\mathrm{Hom}_{\mathbb{C}}(V,W)_{\bar k} = \sum_{\bar i + \bar j \equiv \bar k (\mathrm{mod} 2)} \mathrm{Hom}_{\mathbb{C}}(V_{\bar i}, W_{\bar j})$. We write $\mathrm{End}_{\mathbb{C}}(V)$ for $\mathrm{Hom}_{\mathbb{C}}(V,V)$.

Let $\mathbb{C}^{m|n}$ be the $\mathbb{Z}_2$-graded vector space with even subspace $\mathbb{C}^m$ and odd subspace $\mathbb{C}^n$. Then $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^{m|n})$ with the $\mathbb{Z}_2$-graded commutator forms the general linear superalgebra. To describe its structure, we choose a homogeneous basis $\{v_a \,|\, a \in \mathbf{I}\}$, for $\mathbb{C}^{m|n}$, where $\mathbf{I} = \{1, 2, \ldots, m+n\}$, and v_a is even if $a \le m$, and odd otherwise. The general linear superalgebra relative to this basis of $\mathbb{C}^{m|n}$ will be denoted by $\mathfrak{gl}_{m|n}$. Let E_{ab} be the matrix unit, namely, the $(m+n)\times(m+n)$-matrix with all entries being zero except that at the (a,b) position which is 1. Then $\{E_{ab} \,|\, a, b \in \mathbf{I}\}$ forms a homogeneous basis of $\mathfrak{gl}_{m|n}$, with E_{ab} being even if $a, b \le m$, or $a, b > m$, and odd otherwise. For convenience, we let $\mathbf{I}_1 = \{1, ..., m\}$ and $\mathbf{I}_2 = \{\dot 1, ..., \dot n\}$, where we have written $\dot\nu = \nu + m$. Then $\mathbf{I} = \mathbf{I}_1 \cup \mathbf{I}_2$. Define the map $[\ ] : \mathbf{I} \to \mathbb{Z}_2$ by $[a] = \begin{cases} \bar 0, & \text{if } a \in \mathbf{I}_1, \\ \bar 1, & \text{if } a \in \mathbf{I}_2. \end{cases}$ Now the commutation relations of the general linear superalgebra $\mathfrak{gl}_{m|n}$ can be written as

$$[E_{ab},\ E_{cd}] \;=\; E_{ad}\delta_{bc} - (-1)^{([a]-[b])([c]-[d])} E_{cb}\delta_{ad}.$$

The upper triangular matrices form a Borel subalgebra B of $\mathfrak{gl}_{m|n}$, which contains the Cartan subalgebra H of diagonal matrices. Let $\{\epsilon_a \,|\, a \in \mathbf{I}\}$ be the basis of H^* such that $\epsilon_a(E_{bb}) = \delta_{ab}$. The supertrace induces a bilinear form $(\ ,\ ) : H^* \times H^* \to \mathbb{C}$ on H^* such that $(\epsilon_a, \epsilon_b) = (-1)^{[a]}\delta_{ab}$. Relative to the Borel subalgebra B, the roots of $\mathfrak{gl}_{m|n}$ can be expressed as $\epsilon_a - \epsilon_b,\ a \ne b$, where $\epsilon_a - \epsilon_b$ is even if $[a] + [b] = \bar 0$ and odd

otherwise. The set of the positive roots is $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$, with the set Δ_0^+ of positive even roots and the set Δ_1^+ of the positive odd roots respectively given by

$$\begin{aligned}\Delta_0^+ &= \{\alpha_{i,j} = \epsilon_i - \epsilon_j,\, \alpha_{\nu,\eta} = \epsilon_\nu - \epsilon_\eta \,|\, 1 \le i < j \le m,\, \dot{1} \le \nu < \eta \le \dot{n}\},\\ \Delta_1^+ &= \{\alpha_{i,\nu} = \epsilon_i - \epsilon_\nu \,|\, i \in \mathbf{I}_1,\, \nu \in \mathbf{I}_2\}.\end{aligned}$$

We define a total order on Δ_1^+ by

$$\alpha_{i,\nu} < \alpha_{j,\eta} \quad \Longleftrightarrow \quad \nu - i < \eta - j \text{ or } \nu - i = \eta - j \text{ but } i > j. \tag{2.1}$$

Then $\alpha_{\min} = \alpha_{m,\dot{1}}$, $\alpha_{\max} = \alpha_{1,\dot{n}}$ are respectively the minimal and maximal roots in Δ_1^+.

Throughout the paper, we shall denote by $\mathfrak{g}$ the special linear superalgebra $\mathfrak{sl}_{m|n}$, which is the subalgebra of $\mathfrak{gl}_{m|n}$ consisting of supertraceless matrices. Since $\mathfrak{sl}_{m|n}$ is isomorphic to $\mathfrak{sl}_{n|m}$, we shall assume that $1 \le n \le m$. We choose the Borel subalgebra $\mathfrak{b} = B \cap \mathfrak{g}$ for $\mathfrak{g}$, which contains the Cartan subalgebra $\mathfrak{h} = H \cap \mathfrak{g}$. We identify the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ with the subspace $\sum_{a,b} \mathbb{C}(\epsilon_a - \epsilon_b)$ of H^* spanned by the roots of $\mathfrak{gl}_{m|n}$. Then the roots of $\mathfrak{g}$ coincide with those of $\mathfrak{gl}_{m|n}$, and relative to $\mathfrak{b}$, a root α is positive if and only if $\alpha \in \Delta^+$.

The special linear superalgebra admits a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$, where $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}(m) \oplus \mathbb{C}\check{\rho}_1 \oplus \mathfrak{sl}(n)$ and $\check{\rho}_1$ is defined in (2.6). Also, $\mathfrak{g}_{\pm 1} \subset \mathfrak{g}_{\bar{1}}$, with $\mathfrak{g}_{+1}$ being the nilpotent subalgebra spanned by the odd positive root spaces, and $\mathfrak{g}_{-1}$ that spanned by the odd negative root spaces. A basis of $\mathfrak{g}$ is given by

$$\{E_{ab}, E_{aa} - (-1)^{[b]} E_{bb} \,|\, a, b \in \mathbf{I},\, a \ne b\}.$$

We shall denote

$$\begin{aligned}&e_\alpha = E_{ab}, f_\alpha = E_{ba} \;\text{ if }\; \alpha = \alpha_{a,b} \in \Delta^+ = \Delta_0^+ \cup \Delta_1^+,\\ &h_i = E_{ii} - E_{i+1,i+1},\; h_0 = E_{mm} + E_{\dot{1}\dot{1}},\; h_\nu = E_{\nu\nu} - E_{\nu+1,\nu+1},\end{aligned}$$

for $i \in \mathbf{I}_1 \backslash \{m\}$, $\nu \in \mathbf{I}_2 \backslash \{\dot{n}\}$.

An element in $\mathfrak{h}^*$ is called a weight. A weight λ is integral if $(\lambda, \alpha) \in \mathbb{Z}$ for all roots, and dominant if $2(\lambda, \alpha)/(\alpha, \alpha) \ge 0$ for all positive even roots α of $\mathfrak{g}$. A weight $\lambda \in \mathfrak{h}^*$ can be written in terms of ϵ-basis

$$\lambda = (\lambda_1, ..., \lambda_m \,|\, \lambda_{\dot{1}}, ..., \lambda_{\dot{n}}) = \sum_{a \in \mathbf{I}} \lambda_a \epsilon_a \text{ such that } \sum_{i=1}^{m} \lambda_i + \sum_{\nu=\dot{1}}^{\dot{n}} \lambda_\nu = 0, \tag{2.2}$$

or in terms of Dynkin labels

$$\lambda = [a_1, ..., a_{m-1}; a_0; a_{\dot{1}}, ... a_{\dot{n}-1}],$$

where

$$a_i = \lambda_i - \lambda_{i+1},\; a_0 = \lambda_m + \lambda_{\dot{1}},\; a_\nu = \lambda_\nu - \lambda_{\nu+1},$$

for $i \in \mathbf{I}_1 \backslash \{m\}$, $\nu \in \mathbf{I}_2 \backslash \{\dot{n}\}$. We call λ_p the p-th coordinate of λ for $p \in \mathbf{I}$, and a_p the p-th Dynkin label of λ for $p \in \mathbf{I} \cup \{0\} \backslash \{m, \dot{n}\}$.

The following weights will appear frequently in the remainder of the paper:

$$\mu^{(i,j)} = (\overbrace{0, ..., 0}^{i-1}, j+1, \overbrace{1, ..., 1}^{m-n+j}, 0, ..., 0 \,|\, 0, ..., 0, \overbrace{-1, ..., -1}^{j}, -j-1-m+n, \overbrace{0, ..., 0}^{i-1}), \tag{2.3}$$

for $1 \le i \le n$, $0 \le j \le n+1-i$;

$$\begin{cases} \mu^{(j)} = \mu^{(1,j)}, & \mu^{(j)}_{\pm} = \mu^{(j)} \pm \alpha_{\max},\\ \eta^{(1)} = (1, 1, 0, ..., 0 \,|\, 0, ..., 0, -1, -1), & \eta^{(2)} = (0, ..., 0, -1, -1 \,|\, 1, 1, 0, ..., 0), \end{cases} \tag{2.4}$$

where $\eta^{(1)}$, $\eta^{(2)}$ can occur only when $n \geq 2$;

$$\begin{aligned}
\rho_0 &= \frac{1}{2}\big(\sum_{i=1}^{m}(m-2i+1)\epsilon_i + \sum_{\nu=1}^{n}(n-2\nu+1)\epsilon_{\dot\nu}\big) \\
&= \frac{1}{2}(m-1, m-3, ..., -m+1 \,|\, n-1, n-3, ..., -n+1), && (2.5) \\
\rho_1 &= \frac{1}{2}\big(n\sum_{i=1}^{m}\epsilon_i - m\sum_{\nu=1}^{n}\epsilon_{\dot\nu}\big) = \frac{1}{2}(n, ..., n \,|\, -m, ..., -m), && (2.6) \\
\rho &= \rho_0 - \rho_1,
\end{aligned}$$

where ρ_0 (resp. ρ_1) is half the sum of positive even (resp. odd) roots.

For every integral dominant weight λ, we denote by $L^{(0)}_\lambda$ the finite-dimensional irreducible $\mathfrak{g}_0$-module with highest weight λ. Extend it to a $\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$-module by putting $\mathfrak{g}_{+1}L^{(0)}_\lambda = 0$. Then the Kac module V_λ is the induced module

$$V_\lambda = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_{+1}} L^{(0)}_\lambda \cong U(\mathfrak{g}_{-1}) \otimes_{\mathbb{C}} L^{(0)}_\lambda. \tag{2.7}$$

Denote by L_λ the irreducible module with highest weight λ (which is the unique irreducible quotient module of V_λ) and we always fix a highest weight vector v_λ.

For any finite-dimensional highest weight $\mathfrak{g}$-module V, we can decompose V into a direct sum of $\mathfrak{g}_0$-submodules with respect to its level (an element $x \in \mathfrak{g}$ is said to have level i if $x \in \mathfrak{g}_i$ for $i = -1, 0, 1$, this defines a level structure on V):

$$V = \sum_{\ell\in\mathbb{Z}} V^\ell. \tag{2.8}$$

Set

$$\mathit{top} = \max\{\ell \in \mathbb{Z} \,|\, V^\ell \neq 0\}, \quad \mathit{bottom} = \min\{\ell \in \mathbb{Z} \,|\, V^\ell \neq 0\}.$$

When it is necessary to indicate the module V, we denote them by $\mathit{top}(V)$ and $\mathit{bottom}(V)$. Then $\mathit{top} - \mathit{bottom} \leq mn$. In most cases, we shall specify the highest weight vector v_λ to have level zero, then $\mathit{top} = 0$. But in some cases, we shall shift level so that a vector with weight 0 has level 0.

2.2. **Lie superalgebra cohomology.** In this subsection we explain some basic concepts of Lie superalgebra cohomology. The material can be found in many sources, say, [16]. For $p \geq 1$ and a finite-dimensional $\mathfrak{g}$-module V, let $C^p(\mathfrak{g}, V)$ (the set of p-cochains) be the $\mathbb{Z}_2$-graded vector space of all p-linear mappings φ of $\mathfrak{g}^p = \mathfrak{g} \times \cdots \times \mathfrak{g}$ into V satisfying

$$\varphi(x_1, ..., x_i, x_{i+1}, ..., x_p) = -(-1)^{[x_i][x_{i+1}]}\varphi(x_1, ..., x_{i+1}, x_i, ..., x_p) \text{ (super-skew-symmetry)}$$

for $1 \leq i \leq p-1$ (recall that we denote the degree of an element x by $[x] \in \mathbb{Z}_2$). Set $C^0(\mathfrak{g}, V) = V$. We define the differential operator $d : C^p(\mathfrak{g}, V) \to C^{p+1}(\mathfrak{g}, V)$ by

$$\begin{aligned}
&(d\varphi)(x_0, ..., x_p) \\
&\quad = \sum_{i=0}^{p}(-1)^{i+[x_i]([\varphi]+[x_0]+\cdots+[x_{i-1}])}x_i\varphi(x_0, ..., \hat{x}_i, ..., x_p) \\
&\quad + \sum_{i<j}(-1)^{j+[x_j]([x_{i+1}]+\cdots+[x_{j-1}])}\varphi(x_0, ..., x_{i-1}, [x_i, x_j], x_{i+1}, ..., \hat{x}_j, ..., x_p), && (2.9)
\end{aligned}$$

for $\varphi \in C^p(\mathfrak{g}, V)$ and $x_0, ..., x_p \in \mathfrak{g}$, where the sign $\hat{}$ means that the element below it is omitted. It can be verified that $d^2 = 0$. Set

$$Z^p(\mathfrak{g}, V) = \mathrm{Ker}(d|_{C^p(\mathfrak{g},V)}), \ \ B^p(\mathfrak{g}, V) = \mathrm{Im}(d|_{C^{p-1}(\mathfrak{g},V)}), \ \ H^p(\mathfrak{g}, V) = Z^p(\mathfrak{g}, V)/B^p(\mathfrak{g}, V).$$

Elements in $Z^p(\mathfrak{g},V)$ are called p-cocycles, elements in $B^p(\mathfrak{g},V)$ are p-coboundaries. Two elements of $Z^p(\mathfrak{g},V)$ are said to be cohomologous if their difference lies in $B^p(\mathfrak{g},V)$. For $\varphi\in Z^p(\mathfrak{g},V)$, we denote by $\overline{\varphi}$ its residue class modulo $B^p(\mathfrak{g},V)$. The space $H^p(\mathfrak{g},V)$ is called the p-th cohomology group.

Let U,V,W be three $\mathfrak{g}$-modules such that

$$0\to U\stackrel{f}{\to}V\stackrel{g}{\to}W\to 0 \tag{2.10}$$

is a short exact sequence, where f,g are homogenous $\mathfrak{g}$-module homomorphisms. Then there exists a long exact sequence

$$\cdots\to H^p(\mathfrak{g},U)\stackrel{f^p}{\longrightarrow}H^p(\mathfrak{g},V)\stackrel{g^p}{\longrightarrow}H^p(\mathfrak{g},W)\to H^{p+1}(\mathfrak{g},V)\to\cdots, \tag{2.11}$$

where the maps f^p, g^p can be defined easily from f, g (cf. [16, (2.50)]).

For $x\in\mathfrak{g}$, we define $i_x: C^p(\mathfrak{g},V)\to C^{p-1}(\mathfrak{g},V)$, $\theta_x: C^p(\mathfrak{g},V)\to C^p(\mathfrak{g},V)$ by

$$(i_x\varphi)(x_1,...,x_{p-1})=(-1)^{[x][\varphi]}\varphi(x,x_1,...,x_{p-1}), \tag{2.12}$$

$$\begin{aligned}&(\theta_x\varphi)(x_1,...,x_p)\\&\quad=x\varphi(x_1,...,x_p)\\&\quad-\sum_{i=1}^{p}(-1)^{[x]([\varphi]+[x_1]+\cdots+[x_{i-1}])}\varphi(x_1,...,x_{i-1},[x,x_i],x_{i+1},...,x_p).\end{aligned} \tag{2.13}$$

One can verify that

$$di_x+i_xd=\theta_x,\quad d\theta_x=\theta_xd. \tag{2.14}$$

Note that $\theta:x\mapsto\theta_x$ defines a $\mathfrak{g}$-module structure on $C^p(\mathfrak{g},V)$ such that $Z^p(\mathfrak{g},V)$, $B^p(\mathfrak{g},V)$ are submodules by (2.14). Since $\mathfrak{g}_0$ is a reductive Lie algebra, we can decompose

$$Z^p(\mathfrak{g},V)=Z^p_0(\mathfrak{g},V)\oplus B^p(\mathfrak{g},V) \tag{2.15}$$

as a direct sum of $\mathfrak{g}_0$-submodules. For any $\mathfrak{g}$-module V, we denote

$$V^{\mathfrak{g}_0}=\{v\in V\,|\,\mathfrak{g}_0v=0\}. \tag{2.16}$$

Elements in $V^{\mathfrak{g}_0}$ are called $\mathfrak{g}_0$-invariant elements.

Proposition 2.1. *Every p-cocycle is cohomologous to a $\mathfrak{g}_0$-invariant p-cocycle. More precisely,*

$$Z^p_0(\mathfrak{g},V)\subset Z^p(\mathfrak{g},V)^{\mathfrak{g}_0}. \tag{2.17}$$

Proof. Let $\varphi\in Z^p_0(\mathfrak{g},V)$. For all $x\in\mathfrak{g}_0$, we have, by (2.14),

$$\theta_x\varphi=di_x\varphi+i_xd\varphi=di_x\varphi\in B^p(\mathfrak{g},V)\cap Z^p_0(\mathfrak{g},V)=\{0\}.$$

This proves (2.17). Now the first statement follows from (2.15). □

3. First cohomology groups with coefficients in Kac modules

Let $\varphi\in Z^1(\mathfrak{g},V_\lambda)$ be a 1-cocycle, by Proposition 2.1, we can suppose $\varphi\in Z^1(\mathfrak{g},V_\lambda)^{\mathfrak{g}_0}$. Denote $\varphi^{(0)}=\varphi|_{\mathfrak{g}_0}$. For $x\in\mathfrak{g}_0$, $z\in\mathfrak{g}$, by (2.9) and (2.13), we have

$$0=d\varphi(x,z)=\theta_x\varphi(z)-(-1)^{[z][\varphi]}z\varphi(x)=-(-1)^{[z][\varphi]}z\varphi(x), \tag{3.1}$$

which implies that $\mathfrak{g}$ acts trivially on $\varphi(x)$.

We shall divide the study into two cases according to the highest weights of the Kac modules.

We first assume that $\lambda \neq 2\rho_1$. Then V_λ does not contain a trivial $\mathfrak{g}$-submodule. So, $\varphi^{(0)} = 0$. By (2.8), for any $v \in V_\lambda$, we can uniquely write

$$v = \sum_{\ell=0}^{mn} v_{-\ell}, \quad \text{where} \quad v_{-\ell} \in V_\lambda^{-\ell}.$$

Now (2.9) gives

$$f_\alpha \varphi(f_\beta) + f_\beta \varphi(f_\alpha) = (-1)^{[\varphi]} d\varphi(f_\alpha, f_\beta) = 0 \quad \text{for} \quad \alpha, \beta \in \Delta_1^+. \tag{3.2}$$

As $\mathrm{U}(\mathfrak{g}_{-1})$ is a Grassmann algebra generated by the odd root vectors, and acts freely on the Kac module V_λ (cf. (2.7)), we can take Grassmannian differentiation $\frac{\partial}{\partial f_\alpha}$ with respect to f_α, $\alpha \in \Delta_1^+$, to obtain

$$\varphi(f_\beta) - f_\alpha \tfrac{\partial}{\partial f_\alpha} \varphi(f_\beta) + \delta_{\alpha,\beta} \varphi(f_\alpha) - f_\beta \tfrac{\partial}{\partial f_\alpha} \varphi(f_\alpha) = 0. \tag{3.3}$$

Sum over $\alpha \in \Delta_1^+$, we obtain

$$(mn+1)\varphi(f_\beta) - \sum_{\ell=0}^{mn} \ell \varphi(f_\beta)_{-\ell} = f_\beta v', \quad \text{where} \quad v' = \sum_{\alpha \in \Delta_1^+} \tfrac{\partial}{\partial f_\alpha} \varphi(f_\alpha). \tag{3.4}$$

Note that v' has degree $[\varphi]$. Denote

$$v_{-\ell} = (mn+1-\ell)^{-1} (-1)^{[\varphi]} v'_{-\ell+1}, \text{ and } v = \sum_{\ell=0}^{mn} v_{-\ell}.$$

Then by (3.4),

$$\varphi(f_\beta) = (-1)^{[\varphi]} f_\beta v = dv(f_\beta) \text{ for all } \beta \in \Delta_1^+. \tag{3.5}$$

Replacing v by $v - v_{-mn}$, we still have (3.5), and we can suppose

$$v_{-mn} = 0. \tag{3.6}$$

By (3.5), and the $\mathfrak{g}_0$-invariant property of φ, we have

$$f_\beta x v = x f_\beta v - [x, f_\beta] v = (-1)^{[\varphi]} (x \varphi(f_\beta) - \varphi([x, f_\beta])) = 0 \quad \text{for} \quad x \in \mathfrak{g}_0.$$

This together with equation (3.6) shows that $\mathfrak{g}_0 v = 0$. Thus by replacing φ by $\varphi - dv$, we still have that φ is $\mathfrak{g}_0$-invariant, and

$$\varphi(f_\beta) = 0 \quad \text{for} \quad \beta \in \Delta_1^+. \tag{3.7}$$

Now using the fact that for all $\alpha, \beta \in \Delta_1^+$,

$$0 = (-1)^{[\varphi]} d\varphi(e_\alpha, f_\beta) = e_\alpha \varphi(f_\beta) + f_\beta \varphi(e_\alpha) = f_\beta \varphi(e_\alpha), \tag{3.8}$$

we obtain that either $\varphi(e_\alpha) \equiv 0$, or for some α, $0 \neq \varphi(e_\alpha) \in V_\lambda^{-mn}$ (the bottom level) and $V_\lambda^{-mn} \cong \mathfrak{g}_{+1}$ as $\mathfrak{g}_0$-modules.

In the former case, $H^1(\mathfrak{g}, V_\lambda) = 0$. In the latter case,

$$\lambda = 2\rho_1 + \alpha_{\max} = (n+1, n, ..., n \,|\, -m, ..., -m, -m-1),$$

and we only need to consider such 1-cocycles ϕ that satisfy the conditions

$$\begin{cases} \phi|_{\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}} & = \; 0, \\ \phi(e_\alpha) & = \; \text{the image of } e_\alpha \text{ in } V_\lambda^{-mn} \text{ under the} \\ & \quad\;\, \mathfrak{g}_0\text{-module isomorphism } \mathfrak{g}_{+1} \cong V_\lambda^{-mn}. \end{cases} \tag{3.9}$$

Obviously, ϕ is $\mathfrak{g}_0$-invariant. In fact, all 1-cochains satisfying (3.9) are 1-cocycles. To prove this claim, it only requires to verify the condition

$$e_\alpha \phi(e_\beta) + e_\beta \phi(e_\alpha) = 0 \text{ for all } \alpha, \beta \in \Delta_1^+. \tag{3.10}$$

Since the left hand side of (3.10) is in $V_\lambda^{bottom+1}$, condition (3.10) holds if and only if

$$f_\gamma(e_\alpha\phi(e_\beta)+e_\beta\phi(e_\alpha))=0 \text{ for all } \alpha,\beta,\gamma\in\Delta_1^+. \tag{3.11}$$

The left hand side of (3.11) is equal to

$$\begin{aligned}&[f_\gamma,e_\alpha]\phi(e_\beta)+[f_\gamma,e_\beta]\phi(e_\alpha)-e_\alpha f_\gamma\phi(e_\beta)-e_\beta f_\gamma\phi(e_\alpha)\\ &=\phi([[f_\gamma,e_\alpha],e_\beta]+[[f_\gamma,e_\beta],e_\alpha])=0.\end{aligned} \tag{3.12}$$

Thus ϕ is a 1-cocycle and it is non-trivial because $\phi(e_\alpha)$ is in the bottom level of V_λ which cannot be written in the form $e_\alpha v$ for any $v\in V_\lambda$. Using the $\mathfrak{g}_0$-invariant property of φ, and the fact that $\mathfrak{g}_{+1}$ is irreducible as a $\mathfrak{g}_0$-module, we easily see that the space of all the 1-cocycles satisfying (3.9) is 1-dimensional. Thus

$$H^1(\mathfrak{g},V_\lambda)\cong\mathbb{C}\ \text{ if }\ \lambda=2\rho_1+\alpha_{\max}. \tag{3.13}$$

Note that the above discussion actually provides an explicit construction of the non-trivial 1-cocycles.

Now we consider the remaining case with $\lambda=2\rho_1$. This time the Kac module V_λ is free over $\mathrm{U}(\mathfrak{g}_{-1})$ of rank 1. Let v_λ be a fixed non-zero $\mathfrak{g}$-highest weight vector of V_λ. Then $\mathbb{C}\prod_{\alpha\in\Delta_1^+}f_\alpha v_\lambda$ forms a 1-dimensional $\mathfrak{g}$-submodule of V_λ, where the product $\prod_{\alpha\in\Delta_1^+}f_\alpha$ is ordered with respect to the ordering in (2.1).

Consider any $\varphi\in Z^1(\mathfrak{g},V_\lambda)^{\mathfrak{g}_0}$. By equation (3.1), this φ must satisfy

$$\varphi|_{\mathfrak{g}_0^{\rm ss}}=0 \text{ and } \varphi(2\check\rho_1)=c'\prod_{\alpha\in\Delta_1^+}f_\alpha v_\lambda\in\mathbb{C}\prod_{\alpha\in\Delta_1^+}f_\alpha v_\lambda, \tag{3.14}$$

for some $c'\in\mathbb{C}$, where $\mathfrak{g}_0^{\rm ss}$ denotes the semi-simple part of $\mathfrak{g}_0$. Similar as before, we can suppose that (3.7) holds. Then

$$0=d\varphi(e_\alpha,f_\beta)=(-1)^{[\varphi]}f_\beta\varphi(e_\alpha)-\varphi([e_\alpha,f_\beta])=(-1)^{[\varphi]}f_\beta\varphi(e_\alpha)-\delta_{\alpha,\beta}\varphi(h_\alpha), \tag{3.15}$$

where the last equality follows from (3.14). Note that $h_\alpha=(h_\alpha-\frac{1}{mn}(2\check\rho_1))+\frac{1}{mn}(2\check\rho_1)$ and $h_\alpha-\frac{1}{mn}(2\check\rho_1)\in\mathfrak{g}_0^{\rm ss}$, we thus obtain

$$f_\beta\varphi(e_\alpha)=(-1)^{[\varphi]}\delta_{\alpha,\beta}\frac{1}{mn}\varphi(2\check\rho_1).$$

In particular $f_\beta\varphi(e_a)=0$ for all $\beta\neq\alpha$. Thus $\varphi(e_\alpha)$ has the form $c_\alpha\prod_{\gamma\in\Delta_1^+\backslash\{\alpha\}}f_\gamma v_\lambda$ for some $c_\alpha\in\mathbb{C}$. Combining this with (3.14), we obtain

$$\varphi(e_\alpha)=c'c_\alpha\prod_{\gamma\in\Delta_1^+\backslash\{\alpha\}}f_\gamma v_\lambda, \tag{3.16}$$

where

$$c_\alpha=(-1)^{[\varphi]+m(\alpha)}\frac{1}{mn},\ \text{ and }\ m(\alpha)=\#\{\gamma\in\Delta_1^+\,|\,\gamma<\alpha\}. \tag{3.17}$$

Now if we define a 1-cochain ϕ' by setting

$$\phi'|_{\mathfrak{g}_0^{\rm ss}\oplus\mathfrak{g}_{-1}}=0\ \text{ and }\ \phi'(2\check\rho_1)=\prod_{\gamma\in\Delta_1^+}f_\gamma v_\lambda,\ \ \phi'(e_\alpha)=c_\alpha\prod_{\gamma\in\Delta_1^+\backslash\{\alpha\}}f_\gamma v_\lambda, \tag{3.18}$$

then one can check that ϕ' is a $\mathfrak{g}_0$-invariant 1-cochain (note that $\mathbb{C}v_\lambda$ is a trivial $\mathfrak{g}_0^{\rm ss}$-module). To verify that it is a cocycle, we only need to show that condition (3.10) holds. This is indeed true as follows from (3.12). Thus ϕ' is a 1-cocycle. Furthermore, it is non-trivial, as can be seen from the following arguments: if $\phi'(e_\alpha)$ has the form $e_\alpha v$ for some $v\in V_\lambda$, then v is in the bottom level, i.e., $v\in\mathbb{C}\prod_{\gamma\in\Delta_1^+}f_\gamma v_\lambda$. But $\mathbb{C}\prod_{\gamma\in\Delta_1^+}f_\gamma v_\lambda$

forms a 1-dimensional $\mathfrak{g}$-submodule of V_λ, thus we must have $\phi'(e_\alpha) = e_\alpha v = 0$, a contradiction.

The preceding discussions in this section complete the study of the first cohomology groups with coefficients in Kac modules. We summarize the results below.

Theorem 3.1. *Let V_λ be the finite-dimensional Kac module with highest weight λ. Then*

$$H^1(\mathfrak{g}, V_\lambda) \cong \begin{cases} \mathbb{C} & \textit{if } \lambda = 2\rho_1 + \alpha_{\max}, \\ \mathbb{C} & \textit{if } \lambda = 2\rho_1, \\ 0 & \textit{otherwise.} \end{cases}$$

As an immediate consequence of the theorem, we have the following result.

Corollary 3.2. *Suppose V is any weight module (not necessary finite dimensional) over $\mathfrak{g}$. If V contains a submodule isomorphic to the Kac module $V_{2\rho_1}$, then $H^1(\mathfrak{g},V) \neq 0$.*

Proof. Suppose $v_{2\rho_1} \in V$ is a highest weight vector which generates the Kac module $V_{2\rho_1}$. We define a linear map $\phi' : \mathfrak{g} \to V$ by (3.18). Obviously, ϕ' is a $\mathfrak{g}_0$-invariant 1-cocycle. We claim that it is a non-trivial cocycle. Otherwise $\phi' = dv$ for some $\mathfrak{g}_0$-invariant vector $v \in V$. This in turn leads to

$$\prod_{\alpha\in\Delta_1^+} f_\alpha v_{2\rho_1} = \phi'(2\check{\rho}_1) = 2\check{\rho}_1 v = 0,$$

which is a contradiction. □

Corollary 3.3. $H^1(\mathfrak{g}, U(\mathfrak{g})) \neq 0$.

Proof. Let $v_{2\rho_1} = \prod_{\alpha\in\Delta_1^+} e_\alpha \in U(\mathfrak{g})$ with weight $2\rho_1$. Obviously $v_{2\rho_1}$ is a $\mathfrak{g}$-highest weight vector of $U(\mathfrak{g})$, thus generating a highest weight module V, which is a quotient of the Kac module $V_{2\rho_1}$. We claim that $V = V_{2\rho_1}$. If not, then V does not contain the bottom composition factor (which is the trivial module) of $V_{2\rho_1}$, i.e., $\prod_{\alpha\in\Delta_1^+} \mathrm{ad}\,(f_\alpha) v_{2\rho_1} = 0$, where ad denotes the adjoint action. But we have

$$\prod_{\alpha\in\Delta_1^+} \mathrm{ad}\,(f_\alpha) v_{2\rho_1} = \prod_{\alpha\in\Delta_1^+} \mathrm{ad}\,(f_\alpha) \prod_{\alpha\in\Delta_1^+} e_\alpha = \pm \prod_{\alpha\in\Delta_1^+} h_\alpha + \cdots \neq 0,$$

where $h_\alpha = [e_\alpha, f_\alpha]$. A contradiction. □

4. Second cohomology groups with coefficients in Kac modules

We turn to the computation of the second cohomology groups with coefficients in the Kac modules. By Proposition 2.1, we only need to consider $\mathfrak{g}_0$-invariant 2-cocycles.

Let $\varphi \in Z^2(\mathfrak{g}, V_\lambda)^{\mathfrak{g}_0}$. Since $\mathfrak{g}_0^{\rm ss}$ is semi-simple, we have $H^2(\mathfrak{g}_0^{\rm ss}, V_\lambda) = 0$, i.e., $\varphi|_{\mathfrak{g}_0^{\rm ss}\times\mathfrak{g}_0^{\rm ss}} = d\psi$ for some $\mathfrak{g}_0^{\rm ss}$-invariant 1-cochain $\psi \in C^1(\mathfrak{g}_0^{\rm ss}, V_\lambda)$. Extend ψ to $\psi \in C^1(\mathfrak{g}, V_\lambda)^{\mathfrak{g}_0}$ by setting $\psi|_{\mathfrak{g}_{\bar 1}\oplus\mathbb{C}\check{\rho}_1} = 0$, and replace φ by $\varphi - d\psi$ we can suppose

$$\varphi|_{\mathfrak{g}_0^{\rm ss}\times\mathfrak{g}_0^{\rm ss}} = 0.$$

Then from

$$\begin{aligned} 0 &= d\varphi(x_1, x_2, z) \\ &= \theta_{x_1}\varphi(x_2, z) - \theta_{x_2}\varphi(x_1, z) + (-1)^{[z][\varphi]} z\varphi(x_1, x_2) - \varphi([x_1, x_2], z) \\ &= -\varphi([x_1, x_2], z), \end{aligned}$$

where $x_1, x_2 \in \mathfrak{g}_0^{\rm ss}$, $z \in \mathfrak{g}$, we obtain

$$\varphi|_{\mathfrak{g}_0^{\rm ss}\times\mathfrak{g}} = 0. \tag{4.1}$$

From

$$0 = d\varphi(\check\rho_1, \xi_1, \xi_2) = \theta_{\check\rho_1}\varphi(\xi_1,\xi_2) + (-1)^{[\varphi]}(\xi_1\varphi(\check\rho_1,\xi_2) + \xi_2\varphi(\check\rho_1,\xi_1)) - \varphi(\check\rho_1, [\xi_1,\xi_2]),$$

where $\xi_1, \xi_2 \in \mathfrak{g}_{\bar 1}$, we obtain

$$\xi_1\varphi(\check\rho_1,\xi_2) + \xi_2\varphi(\check\rho_1,\xi_1) = 0, \tag{4.2}$$

(note that $[\xi_1,\xi_2] \in \mathfrak{g}_0 = \mathfrak{g}_0^{\rm ss} \oplus \mathbb{C}\check\rho_1$). This together with equation (4.1) shows that $\psi' : z \mapsto \varphi(\check\rho_1, z)$, $z \in \mathfrak{g}$, is a 1-cocycle.

First assume $\lambda \neq 2\rho_1, 2\rho_1 + \alpha_{\max}$. Then it follows from Theorem 3.1 that ψ' is a 1-coboundary, i.e., there exists $v \in V_\lambda$ of degree $[\varphi]$ such that $\varphi(\check\rho_1, z) = (-1)^{[z][\varphi]}zv$ for $z \in \mathfrak{g}$ and such that $\mathfrak{g}_0 v = 0$ by (4.1). We define a $\mathfrak{g}_0$-invariant 1-cochain ψ of degree $[\varphi]$ by setting $\psi|_{\mathfrak{g}_0^{\rm ss}\oplus\mathfrak{g}_{\bar 1}} = 0$ and $\psi(\check\rho_1) = v$. Then by replacing φ by $\varphi - d\psi$, we have

$$\varphi|_{\mathfrak{g}_0\times\mathfrak{g}} = 0. \tag{4.3}$$

Now $d\varphi(f_\alpha, f_\beta, f_\gamma) = 0$, $\alpha,\beta,\gamma \in \Delta_1^+$, leads to

$$f_\alpha\varphi(f_\beta,f_\gamma) + f_\beta\varphi(f_\alpha,f_\gamma) + f_\gamma\varphi(f_\alpha,f_\beta) = 0 \quad \text{for} \quad \alpha,\beta,\gamma \in \Delta_1^+. \tag{4.4}$$

By taking Grassmannian differentiation $\frac{\partial}{\partial f_\alpha}$ and argue as in the derivations of equations (3.2)–(3.7), we can show that φ is cohomologous to a $\mathfrak{g}_0$-invariant 2-cocycle which satisfies (4.3) and vanishes on $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1}$. Thus we can assume that

$$\varphi(f_\alpha, f_\beta) = 0, \quad \forall\, \alpha,\beta \in \Delta_1^+. \tag{4.5}$$

Under this condition (2.9) gives

$$f_\beta\varphi(e_\alpha,f_\gamma) + f_\gamma\varphi(e_\alpha,f_\beta) = 0, \quad \forall\, \alpha,\beta,\gamma \in \Delta_1^+. \tag{4.6}$$

Again the same arguments as in the derivations of (3.2)–(3.7) renders φ satisfying the following equation

$$\varphi(e_\alpha, f_\beta) = 0, \quad \forall\, \alpha,\beta \in \Delta_1^+. \tag{4.7}$$

Then (4.5) and (4.7) gives

$$f_\alpha\varphi(e_\beta, e_\gamma) = 0. \tag{4.8}$$

Thus $\varphi(e_\alpha, e_\beta)$ is in the bottom level of V_λ. By super-skew-symmetry, $\varphi|_{\mathfrak{g}_{+1}\times\mathfrak{g}_{+1}}$ is in fact a $\mathfrak{g}_0$-invariant map from $\mathfrak{g}_{+1} \wedge \mathfrak{g}_{+1}$ to the bottom level of V_λ. (Here $\wedge$ means symmetric tensor product as $\mathfrak{g}_{+1}$ is odd.) Thus the problem of finding non-trivial 2-cocycles is now reduced to the determination of such maps. Note that

$$\mathfrak{g}_{+1} \wedge \mathfrak{g}_{+1} = L^{(0)}_{2\alpha_{\max}} \oplus L^{(0)}_{\eta^{(1)}}, \text{ as } \mathfrak{g}_0\text{-modules},$$

(cf. notations (2.4)). If $\lambda \neq 2\rho_1 + 2\alpha_{\max}, 2\rho_1 + \eta^{(1)}$, the space of such maps is zero, thus $H^2(\mathfrak{g}, V_\lambda) = 0$.

For the remaining two cases with $\lambda = 2\rho_1 + 2\alpha_{\max}$ and $\lambda = 2\rho_1 + \eta^{(1)}$ respectively, the space S of $\mathfrak{g}_0$-invariant maps from $\mathfrak{g}_{+1} \wedge \mathfrak{g}_{+1}$ to the bottom level of V_λ is 1-dimensional. Let ω be the generator of this space S. Now we can construct a 2-cocycle $\phi_{2,1}$ as follows: set

$$\phi_{2,1}|_{\mathfrak{g}_0\times\mathfrak{g}_{\bar 1}\oplus\mathfrak{g}_{+1}\times\mathfrak{g}_{-1}} = 0, \quad \phi_{2,1}|_{\mathfrak{g}_{+1}\times\mathfrak{g}_{+1}} = \omega. \tag{4.9}$$

Then $H^2(\mathfrak{g}, V_\lambda) = \mathbb{C}\overline{\phi_{2,1}}$.

Next suppose $\lambda = 2\rho_1$. We consider $\varphi \in Z^2(\mathfrak{g}, V_\lambda)^{\mathfrak{g}_0}$ satisfying (4.1). The above arguments show that $\varphi|_{\mathfrak{g}_{+1}\wedge\mathfrak{g}_{+1}} = 0$ and so φ is uniquely determined by $i_{2\check\rho_1}\varphi$, which is a $\mathfrak{g}_0$-invariant 1-cochain. With the help of equation (4.2) we can show that $i_{2\check\rho_1}\varphi$ is

closed. Thus by Theorem 3.1 there exists a constant $c \in \mathbb{C}$ and a $\mathfrak{g}_0$-invariant $v \in V_\lambda$ (v necessarily has weight 0) such that

$$i_{2\check{\rho}_1}\varphi = c\phi' + dv, \tag{4.10}$$

where ϕ' is defined by equation (3.18). As $\varphi(2\check{\rho}_1, 2\check{\rho}_1) = 0$, while $\phi'(2\check{\rho}_1) \neq 0$, the constant c must vanish. Therefore,

$$i_{2\check{\rho}_1}\varphi = dv.$$

We can always express v as $i_{2\check{\rho}_1}\psi$ for some $\mathfrak{g}_0$-invariant 1-cochain ψ such that $d\psi$ satisfies equation (4.1). Therefore,

$$i_{2\check{\rho}_1}(\varphi - d\psi) = 0,$$

which implies that $\varphi = d\psi$. Hence $H^2(\mathfrak{g}, V_{2\rho_1}) = 0$.

Finally we consider the case with $\lambda = 2\rho_1 + \alpha_{\max}$. Any $\varphi \in Z^2(\mathfrak{g}, V_\lambda)^{\mathfrak{g}_0}$ satisfying (4.1) is uniquely determined by $i_{2\check{\rho}_1}\varphi$. We can argue as in the preceding paragraph to show that

$$i_{2\check{\rho}_1}\varphi = c\phi + dv,$$

where $c \in \mathbb{C}$, and ϕ is defined by (3.9). Here $v \in V_\lambda$ is a $\mathfrak{g}_0$-highest weight vector of weight 0. But V_λ does not have such elements (the difference of levels of weight λ and weight 0 is bigger than mn), we have $v = 0$. Thus

$$i_{2\check{\rho}_1}\varphi = c\phi. \tag{4.11}$$

As (4.4) still holds, we can assume that φ satisfies (4.5). Now $d\varphi(e_\alpha, f_\beta, f_\gamma) = 0$, for $\alpha, \beta, \gamma \in \Delta_1^+$, leads to

$$f_\beta\varphi(e_\alpha, f_\gamma) + f_\gamma\varphi(e_\alpha, f_\beta) = \pm\varphi([e_\alpha, f_\beta], f_\gamma) \pm \varphi([e_\alpha, f_\gamma], f_\beta), \tag{4.12}$$

where the sign on the right hand side depends on the parity of φ. Because of (3.9) and (4.1), the right hand side vanishes identically. Thus again we can suppose that equation (4.7) holds. In this case, we have from $d\varphi(f_\alpha, e_\beta, e_\gamma) = 0$, for $\alpha, \beta, \gamma \in \Delta_1^+$, the following equation

$$f_\alpha\varphi(e_\beta, e_\gamma) - (-1)^{[\varphi]}(\varphi([f_\alpha, e_\beta], e_\gamma) + \varphi([f_\alpha, e_\gamma], e_\beta)) = 0. \tag{4.13}$$

Choose a basis $\{e'_\alpha \,|\, \alpha \in \Delta_1^+\}$ for V_λ^{top} such that $\prod_{\tau\in\Delta_1^+} f_\tau e'_\alpha$ is the image of e_α under the $\mathfrak{g}_0$-module isomorphism $\mathfrak{g}_{+1} \cong V_\lambda^{bottom}$. Using (3.9) and (4.1) we can derive from (4.13) the following result

$$\varphi(e_\beta, e_\gamma) = c\big(c_\beta \prod_{\tau\in\Delta_1^+\backslash\{\beta\}} f_\tau e'_\gamma + c_\gamma \prod_{\tau\in\Delta_1^+\backslash\{\gamma\}} f_\tau e'_\beta\big),$$

where c_β, $\beta \in \Delta_1^+$ are the constants defined by equation (3.17). Thus if we define

$$\left\{ \begin{array}{l} \phi_{2,2}|_{\mathfrak{g}_0^{ss}\times\mathfrak{g}\oplus\mathfrak{g}_{-1}\times\mathfrak{g}_{+1}} = 0, \quad i_{2\check{\rho}_1}\phi_{2,2} = \phi, \\ \phi_{2,2}(e_\beta, e_\gamma) = c_\beta \prod_{\tau\in\Delta_1^+\backslash\{\beta\}} f_\tau e'_\gamma + c_\gamma \prod_{\tau\in\Delta_1^+\backslash\{\gamma\}} f_\tau e'_\beta, \end{array} \right. \tag{4.14}$$

we obtain $H^2(\mathfrak{g}, V_\lambda) = \mathbb{C}\overline{\phi_{2,2}}$, as it can be easily shown that $\phi_{2,2}$ is a non-trivial 2-cocycle. Therefore, we obtain the following result.

Theorem 4.1. *For the finite-dimensional Kac module V_λ, we have*

$$H^2(\mathfrak{g}, V_\lambda) = \left\{ \begin{array}{ll} \mathbb{C}\overline{\phi_{2,1}} & \textit{if} \;\; \lambda = 2\rho_1 + 2\alpha_{\max},\ 2\rho_1 + \eta^{(1)}, \\ \mathbb{C}\overline{\phi_{2,2}} & \textit{if} \;\; \lambda = 2\rho_1 + \alpha_{\max}, \\ 0 & \textit{otherwise,} \end{array} \right.$$

where $\phi_{2,1}$, $\phi_{2,2}$ are defined by (4.9) and (4.14). □

5. First cohomology groups with coefficients in irreducible modules

In this section, we compute $H^1(\mathfrak{g}, L_\mu)$ for $\mathfrak{g} = \mathfrak{sl}_{m|n}$, where L_μ is the finite-dimensional irreducible $\mathfrak{g}$-module with highest weight μ. We shall need the notion of primitive vectors, which we recall:

Definition 5.1. *For any $\mathfrak{g}$-module V, a non-zero $\mathfrak{g}_0$-highest weight vector $v \in V$ of weight λ is called a primitive vector and λ a primitive weight if v generates an indecomposable $\mathfrak{g}$-submodule and if there exists a $\mathfrak{g}$-submodule W of V such that $v \notin W$ but $\mathfrak{g}_{+1}v \in W$. If we can take $W = 0$, then v is called a strongly primitive vector or a $\mathfrak{g}$-highest weight vector and λ a strongly primitive weight or a $\mathfrak{g}$-highest weight.*

Let us now begin the computation of the cohomology group. It is known from [3, 2] that $H^1(\mathfrak{g}, \mathbb{C}) = 0$. (This can also be easily proved by a direct computation by using Proposition 2.1.) Thus we suppose $\mu \neq 0$.

Consider an arbitrary $\varphi \in Z^1(\mathfrak{g}, L_\mu)^{\mathfrak{g}_0}$. We shall write

$$\varphi|_{\mathfrak{g}_0} = \varphi^{(0)}, \quad \varphi|_{\mathfrak{g}_{\bar{1}}} = \varphi^{(1)}.$$

Equation (3.1) remains valid in the present case, which implies that the image of $\varphi^{(0)}$ is a $\mathfrak{g}$-submodule of L_μ consisting of invariants. Thus we have $\varphi^{(0)} = 0$.

We now turn to the consideration of $\varphi^{(1)}$. We shall separate this into several cases.

5.1. The case with all $\varphi(f_\alpha) \in L_\mu^{top}$ and some $\varphi(f_\alpha) \neq 0$.

This implies that $\mu = -\alpha_{\min}$. Then from

$$0 = (-1)^{[\varphi]} d\varphi(e_\alpha, f_\beta) = e_\alpha \varphi(f_\beta) + f_\beta \varphi(e_\alpha) = f_\beta \varphi(e_\alpha), \quad \alpha, \beta \in \Delta_1^+,$$

it follows that non-zero $\varphi(e_\alpha)$ must be in the bottom level L_μ^{bottom} of L_μ. This requires the $\mathfrak{g}_0$-highest weight μ_{bottom} of L_μ^{bottom} to be $\alpha_{\max}$, because of the $\mathfrak{g}_0$-invariance of φ. However, by using [23, Proposition 3.5] or [20, Theorem 3.5], we can easily obtain $\mu_{bottom} = (-1, ..., -1, -n \,|\, m, 1, ..., 1) \neq \alpha_{\max}$. Thus $\varphi(e_\alpha) = 0$, for all $\alpha \in \Delta_1^+$.

Let us now define a $\mathfrak{g}_0$-invariant 1-cochain ϕ_1 in the following way: fix a $\mathfrak{g}_0$-module isomorphism $J : \mathfrak{g}_{-1} \cong L_\mu^{top}$, and let

$$\phi_1|_{\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}} = 0, \quad \phi_1(f_\alpha) = J(f_\alpha).$$

To verify that ϕ_1 is a cocycle, we need to show that the following condition

$$f_\alpha \phi_1(f_\beta) + f_\beta \phi_1(f_\alpha) = 0 \text{ for all } \alpha, \beta \in \Delta_1^+,$$

is satisfied, which is equivalent to

$$e_\gamma(f_\alpha \phi_1(f_\beta) + f_\beta \phi_1(f_\alpha)) = 0 \text{ for all } \alpha, \beta, \gamma \in \Delta_1^+. \tag{5.1}$$

Easy manipulations similar to the derivation of (3.12) can show that (5.1) indeed holds. We can also easily show that ϕ_1 is a non-trivial 1-cocycle. Clearly, φ must be a scalar multiple of ϕ. Therefore

$$H^1(\mathfrak{g}, L_\mu) = \mathbb{C}\overline{\phi_1} \text{ if } \mu = -\alpha_{\min}. \tag{5.2}$$

5.2. **The case with all $\varphi(e_\alpha) \in L_\mu^{bottom}$ and some $\varphi(e_\alpha) \neq 0$.** This implies that the $\mathfrak{g}_0$-highest weight μ_{bottom} of L_μ^{bottom} is $\alpha_{\max}$, thus $\mu = \mu^{(n-1)}$ (cf. notations (2.4)) by [23, Proposition 3.5] or [20, Theorem 3.5]. Similar as above, we can construct a non-trivial $\mathfrak{g}_0$-invariant 1-cocycle ϕ_1' in the following way: fix a $\mathfrak{g}_0$-module isomorphism $J' : \mathfrak{g}_{+1} \cong L_\mu^{bottom}$, and set

$$\phi_1'|_{\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}} = 0, \quad \phi_1'(e_\alpha) = J'(e_\alpha). \tag{5.3}$$

Then

$$H^1(\mathfrak{g}, L_\mu) = \mathbb{C}\overline{\phi_1'} \text{ if } \mu = \mu^{(n-1)}. \tag{5.4}$$

5.3. **The remaining case.** Finally we assume that $\varphi(e_\alpha) \notin L_\mu^{bottom}$ and $\varphi(f_\alpha) \notin L_\mu^{top}$ for any non-zero $\varphi(e_\alpha)$ and $\varphi(f_\alpha)$. In particular the given condition implies that $\mu \neq -\alpha_{\min},\ \mu^{(n-1)}$.

In order for $H^1(\mathfrak{g}, L_\mu) \neq 0$, all the central elements of $U(\mathfrak{g})$ contained in $\mathfrak{g}U(\mathfrak{g})$ must act on L_μ by zero [16, Prop. 2.2]. Thus by equations (3.33)–(3.35) of [16], μ has the following form: there exists some k with $0 \leq k \leq n$ such that

$$\begin{aligned} \mu = {} & (\mu_1, ..., \mu_m \,|\, \mu_{\dot 1}, ..., \mu_{\dot n}) \\ = {} & \Big(\mu_1, ..., \mu_k, \overbrace{k, ..., k}^{m-n+j_k}, \overbrace{k-1, ..., k-1}^{j_{k-1}}, ..., \overbrace{1, ..., 1}^{j_1}, \overbrace{0, ..., 0}^{j_0} \Big| \\ & \overbrace{0, ..., 0}^{j_0}, \overbrace{-1, ..., -1}^{j_1}, ..., \overbrace{-k, ..., -k}^{j_k}, -\mu_k - m + n, ..., -\mu_1 - m + n\Big), \end{aligned} \tag{5.5}$$

where

$$\mu_1 \geq ... \geq \mu_k \geq k, \quad j_0, j_1, ..., j_k \geq 0, \quad j_0 + j_1 + \cdots + j_k = n - k. \tag{5.6}$$

If $k = 0$, we shall regard the set $\{\mu_1, ..., \mu_k\}$ as empty, and $\mu = 0$. Since we have assumed $\mu \neq 0$, we have $k \geq 1$.

Needless to say, not all the L_μ with highest weights μ belonging to the list (5.5) have non-trivial first cohomology. We now device ways to eliminate all the weights with trivial first cohomology.

Note that an irreducible module L_μ can always be embedded in a unique Kac module V_{Λ_μ} as the minimal submodule, where Λ_μ is uniquely determined by μ, see [20, Theorem 3.2] or [23, Conjecture 4.1] (which was proved in [1, Main Theorem]). Denote by μ^* the highest weight of the dual module L_μ^* of L_μ. A weight μ is self-dual if $\mu^* = \mu$.

For any weight $\lambda = (\lambda_1, ..., \lambda_m \,|\, \lambda_{\dot 1}, ..., \lambda_{\dot n})$ as in (2.2), we define its level to be

$$\ell(\lambda) = \sum_{i=1}^{m} \lambda_i.$$

Then $top\,(L_\mu) = \ell(\mu)$, $bottom\,(L_\mu) = \ell(\mu_{bottom})$. By [20, Theorem 3.2], [23, Conjecture 4.1] and [1, Main Theorem], $\mu_{bottom} = \Lambda_\mu - 2\rho_1$, thus $bottom\,(L_\mu) = \ell(\Lambda_\mu) - mn$.

There exists an automorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ of $\mathfrak{g}$ which interchanges $\mathbb{C}e_\alpha$'s and $\mathbb{C}f_\alpha$'s:

$$\omega(e_\alpha) = -(-1)^i f_\alpha,\ \omega(f_\alpha) = -e_\alpha,\ \omega(h) = -h \text{ for } \alpha \in \Delta_i^+,\ i \in \mathbb{Z}_2,\ h \in \mathfrak{h}. \tag{5.7}$$

Using this automorphism we may define a new action of $\mathfrak{g}$ on the space L_μ. Under this new action, the module becomes L_μ^*. This in particular implies that $H^p(\mathfrak{g}, L_\mu) \cong$

$H^p(\mathfrak{g}, L^*_\mu)$, $p \geq 0$. Therefore by considering μ^* instead of μ if necessary, we can assume that $top\,(L_\mu) \leq -bottom\,(L_\mu)$. This implies

$$top\,(L_\mu) \leq \frac{1}{2}(mn+1), \quad top\,(L_\mu) + \ell(\Lambda_\mu) \leq mn. \tag{5.8}$$

If $k = n$, then $\mu = (n, ..., n \,|\, -m, ..., -m) = 2\rho_1$, which does not satisfy (5.8). Thus $1 \leq k \leq n-1$. This in particular proves that $H^1(\mathfrak{g}, L_\mu) = 0$ if $n = 1$.

We can now suppose $n \geq 2$. We shall find further conditions on the highest weight μ.

Regard $\mathrm{U}(\mathfrak{g}_{+1})$ as a $\mathfrak{g}_0$-module under the adjoint action. We define a $\mathfrak{g}_0$-module homomorphism $\sigma : U(\mathfrak{g}_{-1}) \to L_\mu$ by

$$\sigma\big(\prod_{\alpha\in S} e_\alpha\big) = \prod_{\alpha\in S\setminus\{\alpha_S\}} e_\alpha \varphi(e_{\alpha_S}), \tag{5.9}$$

where S is any subset of Δ_1^+ with α_S being the largest element, and the product is in the order (2.1). Using $e_\alpha\varphi(e_\beta) = -e_\beta\varphi(e_\alpha)$, one can verify that (5.9) indeed defines a $\mathfrak{g}_0$-module homomorphism, which is non-zero (and thus surjective) so long as $\varphi(e_{\alpha_{max}}) \neq 0$.

By assumption, $\varphi(\mathfrak{g}_{+1}) \not\subset L_\mu^{bottom}$, in particular $\varphi(e_{\alpha_{max}}) \neq 0$, thus the top level L_μ^{top} of L_μ must be a $\mathfrak{g}_0$-submodule of $\sigma(\mathrm{U}(\mathfrak{g}_{+1}))$. This in particular implies that μ coincides with some $\mathfrak{g}_0$ highest weight of $\mathrm{U}(\mathfrak{g}_{+1})$. It is known from [5, 4.1.1] that a $\mathfrak{g}_0$-highest weight in $U(\mathfrak{g}_{+1}) \cong \wedge(\mathfrak{g}_{+1})$ (the exterior algebra of the vector space $\mathfrak{g}_{+1}$) has the following form

$$\mu = (\mu_1, \mu_2, ..., \mu_m \,|\, -\mu'_{\dot n}, -\mu'_{\dot n-1}, ..., -\mu'_{\dot 1}), \tag{5.10}$$

where $(\mu_1, \mu_2, ..., \mu_m)$ is a partition of $\ell(\mu)$ satisfying

$$0 \leq \mu_m \leq ... \leq \mu_1 \leq n, \tag{5.11}$$

and $(\mu'_{\dot 1}, \mu'_{\dot 2}, ..., \mu'_{\dot n})$ is the transpose partition $(\mu_1, \mu_2, ..., \mu_m)^T$ of $(\mu_1, \mu_2, ..., \mu_m)$, i.e.,

$$\mu'_\nu = \#\{i \,|\, 1 \leq i \leq m,\ \mu_i \geq \nu\}. \tag{5.12}$$

Using these conditions, we obtain that μ has the form (5.5) such that (5.6) holds, and

$$\mu_1 = j_1 + \cdots + j_k + k,\ ...,\ \mu_k = j_k + k. \tag{5.13}$$

Note that all such μ have maximal atypicality (i.e., it is n-fold atypical, cf. [16]) and $\alpha_{\min}$ is the first atypical root.

To get further conditions on μ, we need information from Appendix A. The concepts of atypicality matrices, northeast chains (NE) of a weight, and n-, q-, c-relationships of atypical roots, etc. to be used below are all explained in the Appendix.

By (A.4) and Lemma A.6(1), we obtain

$$\ell(\Lambda_\mu) = top + \#NE_\mu \geq top + \#P_\mu = mn + (m+n)\mu_m - top\,.$$

This together with condition (5.8) and Lemma A.6(2) shows that

$$\mu_m = 0 \text{ and } \gamma_1 \text{ is } c\text{- or } q\text{-related to any other atypical roots.} \tag{5.14}$$

When μ has the form (5.5) satisfying (5.6) and (5.13), condition (5.14) becomes

$$j_s + j_{s+1} + \cdots + j_k \leq n - k - s \text{ for } s = 1, ..., k. \tag{5.15}$$

This condition in particular shows that

$$1 \leq k \leq \frac{n}{2}, \tag{5.16}$$

because if $k > \frac{n}{2}$, we would take $s = n - k + 1$ in (5.15), then the left hand side of (5.15) is ≥ 0, but the right hand side is -1, a contradiction.

Let us summarize the preceding discussion into the following Lemma.

Lemma 5.2. *In order for $H^1(\mathfrak{g}, L_\mu) \neq 0$, the highest weight μ of L_μ must be of the form (5.5) and satisfy the conditions (5.6), (5.13), (5.15) and (5.16).*

For convenience, we call such a weight μ a k-fold permissible weight.

Identify a positive odd root $\alpha_{i,\dot\nu}$ with the (i,ν)-position in the atypicality matrix $A(\mu)$. We observe that the number of elements of NE_μ in i-th row is

$$\#\{(i,\nu) \in NE_\mu \,|\, 1 \leq \nu \leq n\} = n - \mu_i - \#\{\nu \,|\, \mu'_{\dot\nu} \geq m+1-i\} = n - \mu_i - \mu_{m+1-i},$$

(recall (5.10) and (5.12)), and the number of elements of NE_μ in ν-th column is

$$\#\{(i,\nu) \in NE_\mu \,|\, 1 \leq i \leq m\} = m - \mu'_{\dot n+1-\nu} - \#\{i \,|\, \mu_i \geq \nu\} = m - \mu'_{\dot n+1-\nu} - \mu'_{\dot\nu}.$$

Thus, from this and (A.4), we have

$$\Lambda_\mu = 2\rho_1 - \mu^R, \tag{5.17}$$

where $\mu^R = (\mu_m, ..., \mu_1 \,|\, -\mu'_{\dot 1}, ..., -\mu'_{\dot n})$ is the reverse of μ. Equation (5.17) shows that the $\mathfrak{g}_0$-highest weight μ_{bottom} of L_μ^{bottom} is $-\mu^R$ and thus the lowest weight of L_μ is $-\mu$. This proves

Lemma 5.3. *Suppose $H^1(\mathfrak{g}, L_\mu) \neq 0$ and $\mu \neq -\alpha_{\min}, \mu^{(n-1)}$. Then L_μ is self-dual, i.e., $\mu^* = \mu$.*

Since $\Lambda_\mu \neq 2\rho_1,\ 2\rho_1 + \alpha_{\max}$ by (5.17), Theorem 3.1 shows that $H^1(\mathfrak{g}, V_{\Lambda_\mu}) = 0$. From the short exact sequence $0 \to L_\mu \to V_{\Lambda_\mu} \to V_{\Lambda_\mu}/L_\mu \to 0$, we obtain a long exact sequence of cohomology groups as in (2.11). In particular, since $H^0(\mathfrak{g}, V_{\Lambda_\mu}) = (V_{\Lambda_\mu})^{\mathfrak{g}} = 0$ (where for a $\mathfrak{g}$-module V, we denote $V^{\mathfrak{g}} = \{v \in V \,|\, \mathfrak{g}v = 0\}$, the set of $\mathfrak{g}$-invariant elements of V), and $H^1(\mathfrak{g}, V_{\Lambda_\mu}) = 0$, we obtain

$$H^1(\mathfrak{g}, L_\mu) \cong H^0(\mathfrak{g}, V_{\Lambda_\mu}/L_\mu) = (V_{\Lambda_\mu}/L_\mu)^{\mathfrak{g}}. \tag{5.18}$$

5.4. **Main result on first cohomology groups.** Now we are ready to prove

Theorem 5.4. *Let L_λ be the finite-dimensional irreducible $\mathfrak{g}$-module with highest weight λ. Then $\dim H^1(\mathfrak{g}, L_\mu) \leq 1$ and $H^1(\mathfrak{g}, L_\mu) \neq 0$ if and only if μ is one of the following $n+1$ weights: $-\alpha_{\min}$, $\mu^{(j)}$, $j = 0, ..., n-1$ (note that $\mu^{(j)}$ is self-dual if $j < n-1$ and $(\mu^{(n-1)})^* = -\alpha_{\min}$).*

Proof. By [23, Conjecture 4.1] and [1, Main Theorem], we have that, the multiplicity of L_λ in the composition series of V_{Λ_μ} is $a_{\Lambda_\mu,\lambda} = [V_{\Lambda_\mu} : L_\lambda] \leq 1$. In particular, taking $\lambda = 0$, we have

$$\dim (V_{\Lambda_\mu}/L_\mu)^{\mathfrak{g}} \leq [V_{\Lambda_\mu}, L_0] \leq 1.$$

We want to use [23, Conjecture 4.1] and [1, Main Theorem] to prove

$$[V_{\Lambda_\mu}, L_0] = 1 \tag{5.19}$$

for the weights listed in the theorem. Recall (2.5) and (2.6), and that the n-fold atypical weight 0 has the following atypical roots

$$\gamma_1^0 = \alpha_{m,\dot 1},\ \gamma_2^0 = \alpha_{m-1,\dot 2}, ..., \gamma_n^0 = \alpha_{m+1-n,\dot n},$$

with the corresponding data (see [23, Conjecture 4.1])

$$(k_1, k_2, ..., k_n) = (m+n-1, m+n-3, ..., m-n+1),$$

where $k_s = \#NE_0(s)$ is the number of the s-th northeast chain $NE_0(s)$ of weight 0 (cf. Definition A.3 in Appendix A). We take

$$\theta=(\theta_1,\theta_2,...,\theta_n)=(\overbrace{1,...,1}^{j_0},0,\overbrace{1,...,1}^{j_1},0,...,0,\overbrace{1,...,1}^{j_{n-k}}).$$

Then

$$\lambda_\theta=\dot d\big(\sum_{s=1}^n\theta_sk_s\gamma^0_s\big)=d\big(\sum_{s:\theta_s=1}\ \sum_{\alpha\in NE_0(s)}\alpha\big)=2\rho_1-(\mu_m,...,\mu_1\,|\,-\mu'_{\dot 1},...,-\mu'_{\dot n})=\Lambda_\mu,$$

where in general for a weight λ, we define $\dot d(\lambda)=d(\lambda+\rho)-\rho$, and $d(\lambda)$ is the unique dominant weight in $W\lambda$ (where W is the Weyl group of $\mathfrak g$). Thus by [23, Conjecture 4.1] and [1, Main Theorem], (5.19) holds.

Let $v_0\in V_{\Lambda_\mu}$ be a primitive vector of weight 0 corresponding to the composition factor L_0. Let $V:=U(\mathfrak g)v_0=U(\mathfrak g_{\bar 1})v_0$. Assume that $\mathfrak g_{+1}v_0\not\subset L_\mu$. Let

$$v_\tau\in U(\mathfrak g_{+1})v_0\backslash L_\mu \text{ be a primitive vector of maximal level with weight } \tau. \tag{5.20}$$

Then τ must also have the form (5.5). Since $\mathfrak g_{+1}U(\mathfrak g_{+1})v_0$ is isomorphic to a quotient of $U(\mathfrak g_{+1})$ as a $\mathfrak g_0$-module, τ also has the form (5.10) satisfying condition (5.12) and (5.13). Thus as proved before, we have

$$\text{either } \tau=\mu^{(n-1)} \text{ or } \tau \text{ is a self-dual weight.} \tag{5.21}$$

Let $-\sigma$ be any anti-primitive weight in $U(\mathfrak g_{-1})v_0$ (i.e., $-\sigma$ is the lowest weight of a composition factor). Then as before, σ is self-dual or $\sigma=\mu^{(n-1)}$. Since the lowest weight of V_{Λ_μ} is $-\mu$, we must have that $\mu-\sigma$ is a sum of distinct positive odd roots. Thus σ has to be a self-dual weight (this also shows that there does not exist a primitive weight λ in $\mathfrak g_{-1}U(\mathfrak g_{-1})v_0$). Let V' be the submodule of V generated by self-dual primitive vectors. Then

$$\mathfrak g_{-1}v_0\subset V' \tag{5.22}$$

First suppose $\tau=\mu^{(n-1)}$. Since there is a unique way to write τ as a sum of distinct positive odd roots, we must up to a non-zero scalar have

$$v_\tau=\prod_{\alpha\in\Gamma_\tau}e_\alpha v_0, \tag{5.23}$$

where $\Gamma_\tau=\{\alpha_{1,\dot n},\alpha_{2,\dot n}...,\alpha_{m,\dot n},\alpha_{m,\dot n-1},...,\alpha_{m,\dot 1}\}$. Note that $\Lambda_\mu-\tau$ can also be uniquely written as a sum of distinct positive odd roots. Thus we can up to scalars uniquely write

$$v_\tau=\prod_{\alpha\in\Gamma'_\tau}f_\alpha v_{\Lambda_\mu}+..., \tag{5.24}$$

where Γ'_τ is the unique subset of Δ_1^+ such that $\Lambda_\mu-\tau=\sum_{\alpha\in\Gamma'_\tau}\alpha$ and where the missing terms are in $U(\mathfrak g_{-1})U(\mathfrak g_0^-)\mathfrak g_0^-v_{\Lambda_\mu}$ (where $\mathfrak g_0^-$ is the negative part of the triangular decomposition of $\mathfrak g_0$). Applying any $f_{\alpha_{i,\dot\nu}}$ to (5.23) for $i>1$ and $\nu<n$, since $f_{\alpha_{i,\dot\nu}}$ commutes with e_α, $\alpha\in\Gamma_\tau$, by (5.22), we see that $f_{\alpha_{i,\dot\nu}}v_\tau\in V'$. Clearly, by (5.24), we have $w=f_{\alpha_{i,\dot\nu}}v_\tau\neq0$ for some $i>1$ and $\nu<n$ (since $\Lambda_\mu\neq2\rho_1$, we must have that Γ'_τ is a proper subset of $\{\alpha_{i,\dot\nu}\,|\,i>1,\,\nu<n\}$). By applying $e_\alpha,\alpha\in\Delta_0^+$ to w until it becomes a $\mathfrak g_0$-highest weight vector, we obtain that $f_{\alpha_{j,\dot\eta}}v_\tau$ is a $\mathfrak g_0$-highest weight vector of weight $\tau-\alpha_{j,\dot\eta}$ in V' for some $j>1$, $\eta<n$. Therefore, there exists some primitive weight σ of V' such that $\sigma-(\tau-\alpha_{j,\dot\eta})$ must be a sum of distinct positive odd roots (cf. [21, Lemma 5.2]), which is impossible, because $\sigma_1\le n-1$ and $\tau_1=n$ and $\sigma-(\tau-\alpha_{j,\dot\eta})=(-1,...\,|\,...)$ (the first coordinate is -1) cannot be a sum of distinct positive odd root. Thus τ is a self-dual weight.

By considering the lowest weights, we see that $\mu-\tau$ is a sum of distinct positive odd roots. Thus

Lemma 5.5. *A weight τ appeared in (5.20) must be a k'-fold permissible weight for some $k' \le k$ and $\mu-\tau$ is a sum of distinct positive odd roots.*

Now we continue the proof of Theorem 5.4 and divide it into 2 cases.

Case 1: Suppose $\mu=\mu^{(j)}$ with $j \le n-2$.

By Lemma 5.5, τ must be 1-fold permissible weight and thus is of the form $\tau=\mu^{(\ell)}$ for some $\ell < j$. Let $V^*_{\Lambda_\mu}$ denote the dual module of the Kac module V_{Λ_μ}. Since $-\mu=(\Lambda_\mu-2\rho_1)^R$ (see (5.17)) is the lowest weight of V_{Λ_μ} and the lowest weight vector can be generated by all vectors, we see that μ is the highest weight of $V^*_{\Lambda_\mu}$ and the highest weight vector can generate every vector, i.e., $V^*_{\Lambda_\mu}=V_\mu$. Since τ is self-dual, we have

$$\tau \text{ is a primitive weight of } V_{\Lambda_\mu} \Longleftrightarrow \tau^*=\tau \text{ is a primitive weight of } V^*_{\Lambda_\mu}=V_\mu. \quad (5.25)$$

We shall again use [23, Conjecture 4.1] and [1, Main Theorem] to determine possible τ such that $a_{\mu,\tau}=[V_\mu,L_\tau]=1$. Since $\tau=\mu^{(\ell)}$, we see that all atypical roots of τ are

$$\gamma^\tau_1=\alpha_{m,\dot 1},\gamma^\tau_2=\alpha_{m-1,\dot 2},...,\gamma^\tau_{n-1}=\alpha_{m-n+2,\dot n-1},\gamma^\tau_n=\alpha_{1,\dot n}, \quad (5.26)$$

with the corresponding data $(k_1,...,k_n)$, $k_s=\#NE_\tau(s)$ (cf. Definition A.3) being

$$\begin{cases} k_s=m+n+1-2s \ \text{ if } \ 1\le s\le n-\ell-2, \\ k_{n-\ell-1}=1, \\ k_s=m+n-1-2s \ \text{ if } \ n-\ell\le s\le n-1, \\ k_n=1. \end{cases} \quad (5.27)$$

Let $\theta=(\theta_1,...,\theta_n)\in\mathbb{Z}_2^n$. We want to prove

$$\lambda_\theta\ne\mu \ \text{ if } \theta_s=1 \text{ for some } s\ne n-\ell-1,n. \quad (5.28)$$

Denote $\overline{\tau}=\mu^{(\ell)}+\sum_{s=1}^n\theta_sk_s\gamma^\tau_s+\rho$. Recall (2.3), we obtain that the $\dot n$-th coordinate of $\overline{\tau}$ is $-(m-n+\ell+1)-k_n+\rho_{\dot n}$, i.e.,

$$\overline{\tau}_{\dot n}=-(m-n+\ell+1)-k_n+\rho_{\dot n},$$

and by (5.26) and (5.27),

$$\overline{\tau}_{\dot s}=\begin{cases} 0-(m+n+1-2s)+\rho_{\dot s} & \text{if } s<n-\ell-1, \text{ or} \\ -1-(m+n-1-2s)+\rho_{\dot s} & \text{if } s\ge n-\ell. \end{cases}$$

Suppose $\overline{\tau}_{\dot n}$, $\overline{\tau}_{\dot s}$ are respectively the $\dot p$-th, $\dot q$-th coordinates of $d(\overline{\tau})$ (the unique dominant weight in the Weyl chamber of $\overline{\tau}$). Then $q<p$ since $\overline{\tau}_{\dot n}<\overline{\tau}_{\dot s}$. Then one can easily see that the $\dot p$-th, $\dot q$-th coordinates of λ_θ are respectively $\overline{\tau}_{\dot n}-\rho_{\dot p}=-(m-n+\ell+1)-k_n-n+p$ ≤ -2 and $\overline{\tau}_{\dot s}-\rho_{\dot q}\le -2$ (note that $\rho_{\dot s}-\rho_{\dot q}=q-s$ by (2.5) and (2.6)). Thus we have (5.28). So, we have proved that if $\lambda_\theta=\mu$, then $\theta=(0,...,0,\theta_{n-\ell-1},0,...,0,\theta_n)$. One can easily check that the latter can happen if and only if $j\ge1$ and $\ell=j-1$ and $\theta_{n-\ell-1}=\theta_n=1$. In particular, if $j=0$, then no τ in (5.20) can occur, i.e., $(V_{\Lambda_\mu}/L_\mu)^{\mathfrak g}=\mathbb{C}$. In this case, Theorem 5.4 follows from (5.18).

So suppose $j\ge1$. Then we just proved that τ, if occurred in (5.20), must be the form $\tau=\mu^{(\ell)}$ with $\ell=j-1$. Consider the Kac module V_μ. Suppose the corresponding

primitive vector of weight τ in V_μ is v'_τ. By (5.25), 0 is also a primitive weight of V_μ, and suppose v'_0 is a corresponding primitive vector. We claim that

$$v'_0 \text{ cannot be generated by } v'_\tau. \tag{5.29}$$

(In fact by a conjecture of [6], v'_τ is generated by v'_0). Take

$$\Lambda=\sum_{s=1}^{j}\mu^{(s,j-s)}=(\overbrace{j+1,...,j+1}^{m-n+j+1},0,...,0\,|\,0,...,0,\overbrace{n-m-j-1,...,n-m-j-1}^{j+1}). \tag{5.30}$$

By [21], we see that μ and 0 are strongly primitive weights of V_Λ corresponding to unlinked codes. Suppose v''_μ and v''_0 are their corresponding strongly primitive vectors in V_Λ.

We prove that v''_0 can be generated by v''_μ as follows (in fact, the code corresponding to v''_μ is a sub-code of the code corresponding to v''_0 (see [20, Theorem 3.7])): Consider the dual module $V^*_\Lambda = V_\Sigma$, where $\Sigma = 2\rho_1 - \Lambda^R$. Note that $L^*_\mu = L_\mu$, $L^*_0 = L_0$ and L^*_Λ all are composition factors of V_Σ. Let v_Σ be the highest weight vector of V_Σ. Let $v_{\mu_{bottom}} \in V_\Sigma$ be up to scalars the unique $\mathfrak{g}_0$-highest weight vector of weight $\mu_{bottom} = -\mu^R$ corresponding to the lowest $\mathfrak{g}_0$-highest weight vector of the composition factor L_μ of V_Σ. First note that

$$\Sigma - 0 = 2\rho_1 - \Lambda^R = \sum_{\alpha\in\Gamma_1}\alpha, \tag{5.31}$$

where Γ_1 is a unique subset of Δ_1^+. Thus according to [21, Lemma 5.2], we can up to scalars uniquely write v'''_0 (a primitive vector of weight 0 in V_Σ) as

$$v'''_0 = \prod_{\alpha\in\Gamma_1} f_\alpha v_\Sigma + \cdots, \tag{5.32}$$

where the missing terms are contained in $U(\mathfrak{g}_{-1})U(\mathfrak{g}_0^-)\mathfrak{g}_0^- v_\Sigma$. As in [21], we shall call the first term of the right hand side of (5.32) the leading term of v'''_0. Also note that

$$0 - \mu_{bottom} = \mu^R = \sum_{\alpha\in\Gamma_2}\alpha,$$

for a unique subset Γ_2 of Δ_1^+. As in [21, §5] (where some operators χ_J are defined), we can construct a weight vector $v'_{\mu_{bottom}}$ of weight μ_{bottom} from v'''_0 such that

$$v'_{\mu_{bottom}} = \prod_{\alpha\in\Gamma_2} f_\alpha v'''_0 + \cdots, \tag{5.33}$$

and such that $v'_{\mu_{bottom}}$ is a $\mathfrak{g}_0$-highest weight vector as long as $v'_{\mu_{bottom}} \neq 0$. We claim that $v'_{\mu_{bottom}}$ is indeed non-zero: Substitute (5.32) into (5.33), we see that $v'_{\mu_{bottom}}$ has the leading term $\prod_{\alpha\in\Gamma_1\cup\Gamma_2} f_\alpha v_\Sigma$. This is because: for any $\alpha = \alpha_{i,\dot p} \in \Gamma_1$ and any $\beta = \alpha_{j,\dot q} \in \Gamma_2$, we have either $i > j$ or $p < q$. Thus when we substitute (5.32) into (5.33), we only produce one leading term. Thus the $\mathfrak{g}_0$-highest weight vector $v_{\mu_{bottom}} = v'_{\mu_{bottom}}$ of weight μ_{bottom} in V_Σ can be generated by v'''_0. By taking dual, we obtain that v''_0 in V_Λ can be generated by v''_μ.

By [23, Conjecture 4.1] and [1, Main Theorem], we can prove that $\tau = \mu^{(\ell)}$ is not a primitive weight of V_Λ: Suppose $\Lambda = \lambda_\theta$ for some θ, i.e., $\Lambda+\rho = d(\tau+\sum_{s=1}^n k_s\theta_s\gamma_s^\tau+\rho)$. This means that two sets $\{\Lambda_{\dot s} + \rho_{\dot s} \,|\, s = 1,...,n\}$ and $\{\tau_{\dot s} + k_s\theta_s + \rho_{\dot s} \,|\, s = 1,...,n\}$ are equal, and so by (2.5), (5.30) and (5.27), two sets

$$\{1, 2, ..., n-j-1, m+1, m+2, ..., m+j+1\} \quad \text{and}$$

$$\{\theta_1(m+n-1)+1, \theta_2(m+n-3)+2, ..., \theta_{n-j-1}(m-n+2j+3)+n-j-1,$$
$$\theta_{n-j}+n-j,$$
$$1+\theta_{n-j+1}(m-n+2j-3)+n-j+1, ..., 1+\theta_{n-1}(m-n+1)+n-1,$$
$$m-n+j+\theta_n+n\}$$

are equal. No matter whether $\theta_{n-j}=0$ or 1, the first set does not contain $\theta_{n-j}+n-j$. Thus τ is not a primitive weight of Λ.

Now we define the module homomorphism $\pi: V_\mu \to u(\mathfrak{g})v''_\mu \subset V_\Lambda$ from Kac module V_μ to the highest weight module $U(\mathfrak{g})v''_\mu$ as follows: π sends v_μ to v''_μ. Then π sends v'_τ to zero (since τ is not a primitive weight of V_Λ) but sends v'_0 to v''_0 (since v''_0 is generated by v''_μ, there is a pre-image v'_0 of v''_0 which is also primitive in V_Λ). This proves (5.29).

Now consider $V^*_\mu = V_{\Lambda_\mu}$. By (5.29), we see that in V_{Λ_μ}, v_τ cannot be generated by v_0. This proves that no τ in (5.21) can occur in (5.20), i.e., $(V_{\Lambda_\mu}/L_\mu)^{\mathfrak{g}}=\mathbb{C}$. By (5.18), this proves Theorem 5.4 in this case.

Case 2: Suppose μ is a k-fold permissible weight with $k\geq 2$ (see Remark 5.7 below).

Note that we can uniquely decompose μ as $\mu=\nu+\eta$, where

$$\nu=(\mu_1,\overbrace{1,...,1}^{\mu'_1-1},0,...,0\,|\,0,...,0,\overbrace{-1,...,-1}^{\mu_1-1},-\mu'_1)$$

is a 1-fold permissible weight and $\eta=\mu-\nu$ is a $(k-1)$-fold permissible weight when restricting it to be a dominant weight of $sl(m-1/n-1)$ (η is not dominant as a weight of $\mathfrak{g}$). Using [23, Conjecture 4.1] and [1, Main Theorem], we see that both ν and 0 are primitive weights of V_μ, which correspond to linked codes (cf. [6]). Let w_ν and w_0 be the corresponding primitive vectors in V_μ. Using exactly the same arguments in the paragraph after (5.30), we obtain that w_0 can be generated by w_ν. This means that in $V^*_\mu=V_{\Lambda_\mu}$, a primitive vector of weight ν can be generated by a primitive vector of weight 0, i.e., L_0 is not a submodule in V_{Λ_μ}/L_μ. Thus $(V_{\Lambda_\mu}/L_\mu)^{\mathfrak{g}}=0$. By (5.18), this completes the proof of Theorem 5.4. $\square$

Remark 5.6. *It is also possible to prove Theorem 5.4 using the machinery to be developed in Section 6).*

Remark 5.7. *Suppose μ is a weight such that all the central elements of $U(\mathfrak{g})$ contained in $\mathfrak{g}U(\mathfrak{g})$ act trivially on L_μ. As in (6.28), one can prove that L_μ contains a copy of $\mathfrak{g}_0$-module $\mathfrak{g}_{+1}$ or $\mathfrak{g}_{-1}$ if and only if $\mu=-\alpha_{\min}$, $\mu^{(j)}$, $0\leq j\leq n-1$ (this also shows that $H^1(\mathfrak{g},L_\mu)=0$ if μ is a k-fold permissible weight with $k\geq 2$). Thus*

$$H^1(\mathfrak{g},L_\mu)\neq 0 \iff L_\mu \text{ contains a copy of } \mathfrak{g}_0\text{-module } \mathfrak{g}_{+1} \text{ or } \mathfrak{g}_{-1}. \tag{5.34}$$

6. Second cohomology groups with coefficients in irreducible modules

In this section we compute the second cohomology groups of the special linear superalgebra with coefficients in the finite dimensional irreducible modules.

6.1. **Primitive weight graphs.** We first introduce some concepts, which will be used extensively in the remainder of the paper.

Definition 6.1. *For a $\mathfrak{g}$-module V, we denote by $P(V)$ the set of primitive weights of V. Let $P_0(V)=\{\mu\in P(V)\,|\,L_\mu$ is an irreducible submodule of $V\}$. A weight in $P_0(V)$ is called a lowest or bottom primitive weight. Let $P'(V)=\{v_\mu\,|\,\mu\in P(V)\}$ be a collection of non-zero primitive vectors. For a primitive weight μ of V, we denote*

$U(\mu)=U_V(\mu)=U(\mathfrak{g})v_\mu$. If $\mu\neq\nu$ and $v_\nu\in U_V(\mu)$, we say that ν is derived from μ and write $\nu \twoheadleftarrow\!\!\leftarrowtail \mu$ or $\mu \rightarrowtail\!\!\twoheadrightarrow \mu$. If $\mu \rightarrowtail\!\!\twoheadrightarrow \nu$ and there exists no λ such that $\mu \rightarrowtail\!\!\twoheadrightarrow \lambda \rightarrowtail\!\!\twoheadrightarrow \nu$, then we say ν is directly derived from μ and write $\mu \rightarrowtail \nu$ or $\nu \leftarrowtail \mu$. If $\mu \rightarrowtail \nu$ and $v_\nu\in U(\mathfrak{g}_{+1})v_\mu$, we also use $\mu \overset{e}{\rightarrowtail} \nu$ or $\nu \overset{e}{\leftarrowtail} \mu$ to denote this fact; similarly, we also use $\mu \overset{f}{\rightarrowtail} \nu$ or $\nu \overset{f}{\leftarrowtail} \mu$ to denote $v_\nu\in U(\mathfrak{g}_{-1})v_\mu$ (Sometimes for convenience, we also use symbols $\mu \longleftrightarrow \lambda$ to denote $\mu \rightarrowtail \lambda$ or $\mu \leftarrowtail \lambda$).

We can associate $P(V)$ with a directed graph, still denoted by $P(V)$, called the primitive weight graph of V, such that two weights λ and μ are connected by a single directed edge (i.e., the two weights are linked) pointing toward μ if and only if μ is directly derived from λ. A subgraph of $P(V)$ is a subset S of $P(V)$ together with all edges linking elements of S. A subgraph S of $P(V)$ is closed if it satisfies the following condition: For any $\eta\in P(V)$, $\mu,\nu\in S$,

$$\mu \rightarrowtail\!\!\twoheadrightarrow \eta \rightarrowtail\!\!\twoheadrightarrow \nu \qquad \textit{implies} \qquad \eta\in S.$$

It is clear that a module is indecomposable if and only if its primitive weight graph is connected (in the usual sense). It is also clear that a subgraph of $P(V)$ corresponds to a subquotient of V if and only if it is closed. Thus a subgraph with only 2 weights is always a closed subgraph.

For any subset S (not necessarily a subgraph), we denote by $\overline{S}$ the smallest closed subgraph which contains S.

For any graph Γ, we denote by $M(\Gamma)$ any module with primitive weight graph Γ if it is indeed a primitive weight graph of a module. If Γ is a subgraph of $P(V)$, then $M(\Gamma)$ is defined, and we shall also denote $\overline{M}(\Gamma)=M(\overline{\Gamma})$. If Γ corresponds to a submodule of V and we need to indicate $M(\Gamma)$ as a subquotient module of V, we denote $M(\Gamma)$ by $M_V(\Gamma)$.

For a dominant weight μ, we let $P(\mu)=P(V_\mu)$, and set

$$P^\vee(\mu)=\{\lambda\,|\,\mu\in P(\lambda)\}.$$

Then $P^\vee(\mu)$ is the set of dominant weights λ such that every Kac module V_λ has a composition factor L_μ.

We shall say that a weight μ has non-zero 1-cohomology if $H^1(\mathfrak{g},L_\mu)\neq 0$.

Remark 6.2. (1) *Since we will frequently need to determine primitive weights of Kac modules, it will be convenient to first use permissible codes defined in* [6] *to find a possible primitive weight, then use* [23, *Conjecture* 4.1] *and* [1, *Main Theorem*] *to check if it is a primitive weight.*

(2) *Let $P(V)$ be the primitive weight graph of the module V. The dual primitive weight graph $P^*(V)$ of V is the graph obtained from $P(V)$ by reversing the directions of all arrows and changing all weights to their dual weights. Note that $P^*(V)=P(V^*)$, where V^* denote the dual module of V. If we change the action of $\mathfrak{g}$ on $P(V^*)$ by the automorphism ω defined in (5.7) which exchanges $\mathbb{C}e_\alpha$'s and $\mathbb{C}f_\alpha$'s, then we obtain another module, called the inverse module of V, with graph $\widetilde{P}(V)$ obtained from $P(V)$ by reversing the directions of all arrows (recall that using the automorphism ω, the module L_μ becomes $L_{\mu^*}=L_\mu^*$ for all μ). In particular, we have*

$$\exists\, M(\mu\rightarrowtail\nu) \iff \exists\, M(\mu\leftarrowtail\nu) \iff \exists\, M(\mu^*\rightarrowtail\nu^*) \iff \exists\, M(\mu^*\leftarrowtail\nu^*). \qquad (6.1)$$

(3) *If (6.1) occurs and $\mu\neq\nu$, then $\nu\in P(\mu)\cup P^\vee(\mu)$ since either $M(\mu\rightarrowtail\nu)$ or $M(\mu\leftarrowtail\nu)$ must be a highest weight module (we adopt the convention that a highest weight module is cyclically generated by a highest weight vector), and similarly,*

$\nu^* \in P(\mu^*) \cup P^\vee(\mu^*)$ (*note that it is possible that* $\mu = \nu$, *in this case*, $M(\mu \rightarrowtail \mu)$ *is not necessarily a weight module*). *From this we obtain that if* μ *is a primitive weight of the highest* (*resp. lowest*) *level in an indecomposable module* $P(V)$ *such that either every other primitive weight derives* μ *or is derived from* μ, *then*

$$P(V) \subset P(\mu) \quad (\textit{resp.} \quad P(V) \subset P^\vee(\mu)\,).$$

6.2. **Technical lemmas.** This subsection contains a series of lemmas which will be used in establishing Theorem 6.18 in the next subsection. The proofs of most of the lemmas rely on detailed analysis of structures of Kac modules, which unfortunately is a matter of a very technical nature.

Lemma 6.3. $\mu \longleftrightarrow 0$ *implies that* $H^1(\mathfrak{g}, L_\mu) \cong \mathbb{C}$.

Proof. Say we have a module $V = M(0 \rightarrowtail \mu)$. Then we can define a 1-cocycle $\varphi \in Z^1(\mathfrak{g}, L_\mu)$ by $\varphi(x) = xv_0 \in L_\mu$ for $x \in \mathfrak{g}$, where v_0 is a primitive vector with weight 0. Clearly it is non-trivial, otherwise V is decomposable. □

Lemma 6.4. (1) *If there is a short exact sequence* $0 \to \mathbb{C} \to V \to W \to 0$, *then* $H^1(\mathfrak{g}, V) \cong H^1(\mathfrak{g}, W)$.

(2) *If there is a non-split short exact sequence* $0 \to W \to V \to \mathbb{C} \to 0$, *then* $\dim H^1(\mathfrak{g}, V) = \dim H^1(\mathfrak{g}, W) - 1$.

Proof. (1) Since $H^1(\mathfrak{g}, \mathbb{C}) = H^2(\mathfrak{g}, \mathbb{C}) = 0$, by (2.11), we have

$$0 = H^1(\mathfrak{g}, \mathbb{C}) \to H^1(\mathfrak{g}, V) \to H^1(\mathfrak{g}, W) \to H^2(\mathfrak{g}, \mathbb{C}) = 0,$$

which gives $H^1(\mathfrak{g}, V) \cong H^1(\mathfrak{g}, V)$.

(2) Since the short exact sequence is non-split, we have the exact sequence

$$H^0(\mathfrak{g}, W) \xrightarrow{i} H^0(\mathfrak{g}, V) \xrightarrow{j} H^0(\mathfrak{g}, \mathbb{C}) \to H^1(\mathfrak{g}, W) \to H^1(\mathfrak{g}, V) \to H^1(\mathfrak{g}, \mathbb{C}),$$

where i is the identity map, and so j is the zero map. Thus we have an exact sequence $0 \to \mathbb{C} \to H^1(\mathfrak{g}, W) \to H^1(\mathfrak{g}, V) \to 0$, which gives the result. □

Lemma 6.5. *Suppose* $\mathfrak{g} \neq sl(2/1)$, *and suppose there is a short exact sequence of* $\mathfrak{g}$-*modules*

$$0 \to \mathbb{C} \to V \to W \to 0 \quad \textit{or} \quad 0 \to W \to V \to \mathbb{C} \to 0.$$

Then $H^2(\mathfrak{g}, V) \cong H^2(\mathfrak{g}, W)$.

Proof. Note that $H^3(\mathfrak{g}, \mathbb{C}) = 0$ as long as $\mathfrak{g} \neq sl(2/1)$. We have the following exact sequence (cf. (2.11))

$$\begin{aligned} &0 = H^2(\mathfrak{g}, \mathbb{C}) \to H^2(\mathfrak{g}, V) \to H^2(\mathfrak{g}, W) \to H^3(\mathfrak{g}, \mathbb{C}) = 0 \ \text{ or} \\ &0 = H^1(\mathfrak{g}, \mathbb{C}) \to H^2(\mathfrak{g}, W) \to H^2(\mathfrak{g}, V) \to H^2(\mathfrak{g}, \mathbb{C}) = 0, \end{aligned}$$

which give the result. □

Lemma 6.6. *For any* (*finite-dimensional*) *module* V, *we have*

$$\dim H^1(\mathfrak{g}, V) \le \sum_{\mu \in P(V)} \dim H^1(\mathfrak{g}, L_\mu). \tag{6.2}$$

More generally, suppose

$$P(V) = \Big(\bigcup_{i=1}^{p} P_i\Big) \bigcup \Big(\bigcup_{j=1}^{q} Q_j\Big) \tag{6.3}$$

is a disjoint union such that all P_i, Q_j are closed subgraph of $P(V)$ and such that all Q_j do not contain any primitive weight with non-zero 1*-cohomology, then*

$$\dim H^1(\mathfrak{g},V) \le \sum_{i=1}^{p} \dim H^1(\mathfrak{g},M(P_i)). \tag{6.4}$$

Proof. We prove (6.2) by induction on number of composition factors of V. If V is irreducible, the claim is obvious. Suppose V is not irreducible, and let L_ν be an irreducible submodule of V. Then the exact sequence $0 \to L_\nu \to V \to V/L_\nu \to 0$ gives

$$H^1(\mathfrak{g},L_\nu) \xrightarrow{i} H^1(\mathfrak{g},V) \xrightarrow{j} H^1(\mathfrak{g},V/L_\nu).$$

Thus as a vector space, $H^1(\mathfrak{g},V) \cong i(H^1(\mathfrak{g},L_\nu)) \oplus j(H^1(\mathfrak{g},V))$, and $\dim H^1(\mathfrak{g},V) \le \dim H^1(\mathfrak{g},L_\nu) + \dim H^1(\mathfrak{g},V/L_\nu) \le \sum_{\mu\in P(V)} \dim H^1(\mathfrak{g},L_\mu)$. □

Lemma 6.7. *Suppose V is a module without any trivial* (*i.e.,* 1*-dimensional*) *composition factor. Then*

$$\sum_{\mu\in P_0(V)} \dim H^1(\mathfrak{g},L_\mu) \le \dim H^1(\mathfrak{g},V).$$

Furthermore, suppose we have (*6.3*) *such that all P_i correspond submodules of V and $V/\oplus_{i=1}^{p} M(P_i)$ does not contain a trivial composition factor, then*

$$\sum_{i=1}^{p} \dim H^1(\mathfrak{g},M(P_i)) \le \dim H^1(\mathfrak{g},V).$$

Proof. Let $V' = \oplus_{\mu\in P_0(V)} L_\mu \subset V$. Clearly, $H^1(\mathfrak{g},V') \cong \oplus_{\mu\in P_0(V)} H^1(\mathfrak{g},L_\mu)$. From $0 \to V' \to V \to V/V' \to 0$, by (2.11), we have $0 = H^0(V/V') \to H^1(V') \xrightarrow{i} H^1(V)$, i.e., i is an injective map. □

We need to use [23, Conjecture 4.1] and [1, Main Theorem] to determine $P^\vee(\mu)$ and $P(\mu)$ for some μ. Note that the necessary and sufficient condition for the existence of an element d in the Weyl group of $\mathfrak{g}_0$, which relates the weight $\mu = (\mu_1,...,\mu_m \,|\, \mu_{\dot 1},...,\mu_{\dot n})$ to a weight $\lambda = (\lambda_1,...,\lambda_m \,|\, \lambda_{\dot 1},...,\lambda_{\dot n})$ such that $\lambda+\rho$ is dominant through the equation $\lambda+\rho = d(\mu+\rho)$, is the following equalities of sets:

$$\{\lambda_1{+}m,...,\lambda_m{+}1\} = \{\mu_1{+}m,...,\mu_m{+}1\}, \quad \{\lambda_{\dot 1}{-}1,...,\lambda_{\dot n}{-}n\} = \{\mu_{\dot 1}{-}1,...,\mu_{\dot n}{-}n\},$$

where dominance of $\lambda+\rho$ means that

$$\lambda_1+m \ge \lambda_2+m-1 \ge ... \ge \lambda_m+1, \quad \lambda_{\dot 1}-1 \ge \lambda_{\dot 2}-2 \ge ... \ge \lambda_{\dot n}-n.$$

Recall notations (2.3) and (2.4). For $\tau = \mu^{(\ell)}$, we have (5.26) and (5.27).

Lemma 6.8. (1) *Suppose* $0 \le \ell \le n-1$. *Then*

$$\begin{aligned} P^\vee(\mu^{(\ell)}) = \{\lambda_\theta(\mu^{(\ell)}) = {}& \mu^{(1,j_1)} + \cdots + \mu^{(s-1,j_{s-1})} + \mu^{(s,\ell)} + \mu^{(s+1,j_{s+1})} + \cdots + \mu^{(k,j_k)} \\ & + \theta_{n-\ell-1}\alpha_{m-n-i-\ell+3,\dot n-\ell-i} + \theta_n \alpha_{m+1-i,\dot n+1-i} \;|\; 1 \le s \le n-\ell-1, \\ & s \le k \le n-1,\; j_1 > j_2 > ... > j_{s-1} > \ell+1,\; \ell > j_{s+1} > ... > j_k, \\ & \theta_{n-\ell-1},\theta_n \in \{0,1\}\}, \end{aligned} \tag{6.5}$$

(*note that when $\theta_{n-\ell-1} = \theta_n = 1$, the sum of the last two terms together with $\mu^{(s,\ell)}$ is equal to $\mu^{(s,\ell+1)}$*).

(2) *Similarly, we have*

$$\begin{aligned} P^\vee(-\alpha_{\min}) = \{\lambda_\theta(-\alpha_{\min}) = {}& -\alpha_{\min} + \mu^{(1,j_1)} + \cdots + \mu^{(k,j_k)} + \theta_1\alpha_{\min} \\ & |\; 0 \le k \le n-1, n-1 > j_1 > ... > j_k \ge 0,\; \theta_1 = 0,1\}. \end{aligned} \tag{6.6}$$

Proof. (1) Suppose $\{j\,|\,\theta_j=1,\, j\neq n-\ell-1,n\}=\{j_1,...,j_{s-1},j_{s+1},...,j_k\}$, where $j_1>...>j_{s-1}>\ell+1$, $\ell>j_{s+1}>...>j_k$. Using (5.27), by induction on $\#\{j\,|\,\theta_j=1\}$, we obtain (6.5). Similarly we have (6.6). □

For a weight τ, we denote

$$P_+(\tau)=\big\{\mu\in P(\tau)\,\big|\sum_{i:\,1\le i\le m,\mu_i<0}\mu_i\ge -1\big\},\quad P_-(\tau)=P(\tau)\backslash P_+(\tau).$$

Lemma 6.9. (1) *Suppose* $0\le\ell\le n-1$. *We have*

$$\begin{aligned}P_+(\mu^{(\ell)}) \;=\; &\{\mu^{(\ell)},\ \mu_-^{(\ell)},\ \mu_+^{(\ell-1)},\ \mu^{(\ell-1)},\ 0,\ \mu^{(\ell)}-\alpha_{\min},\\ &\mu_-^{(\ell)}-\alpha_{\min},\ \mu_+^{(\ell-1)}-\alpha_{\min},\ \mu^{(\ell-1)}-\alpha_{\min},\ -\alpha_{\min}\},\end{aligned}\tag{6.7}$$

where if $\ell=n-1$ *then the last 5 weights do not occur, and if* $\ell=0$ *then the 6 weights with plus or minus subscript or with supscript* $(\ell-1)$ *do not occur.*

(2) *If* $\mu\in P_-(\mu^{(\ell)})$, *there is no link* $\mu^{(\ell)}\longleftrightarrow\mu$ *unless* $n\ge 2$, $\ell=n-2$ *and* $\mu=\eta^{(2)}$.

(3) *If* $\mu\in P_-(-\alpha_{\min})$, *there is no link* $-\alpha_{\min}\longleftrightarrow\mu$ *unless* $n\ge 2$, $\mu=-2\alpha_{\min},\eta^{(2)}$.

Proof. (1) Let $\mu\in P_+(\mu^{(\ell)})$. Then μ satisfies the condition (5.5) and so

$$0\le\mu_1\le\ell{+}1,\quad 0\le\mu_2,...,\mu_{m-n+\ell}\le 1,\quad \mu_{m-n+\ell+1}{=}...{=}\mu_{m-1}{=}0,\quad \mu_m=0,-1.$$

From this and using [23, Conjecture 4.1] or [1, Main Theorem], it is straightforward to show that μ must be one of weights in (6.7).

(2) Suppose $\mu\in P_-(\mu^{(\ell)})$. Say $\mu^{(\ell)}\rightarrowtail\mu$. If $\mu\ne-2\alpha_{\min},\eta^{(2)}$, then $H^2(\mathfrak g,L_\mu)=0$ (since μ cannot be a weight in (6.19) either). Since μ must have the form (5.5), one can easily check that the lowest weight of L_μ (which is lower than $-\mu^{(\ell)}$) is not a sum of distinct negative odd roots, this means that $H^1(\mathfrak g,M(\mu^{(\ell)}\rightarrowtail\mu))=0$ (if $H^1(\mathfrak g,M(\mu^{(\ell)}\rightarrowtail\mu))\ne 0$, then in particular the lowest weight vector of L_μ can be generated by a primitive vector of weight 0). Then from the short exact sequence $0\to L_\mu\to M(\mu^{(\ell)}\rightarrowtail\mu)\to L_{\mu^{(\ell)}}\to 0$, by (2.11), we obtain that $H^2(\mathfrak g,L_\mu)\ne 0$, contradicting the fact that $H^2(\mathfrak g,L_\mu)=0$. If $\mu=-2\alpha_{\min},\eta^{(2)}$, then only the second case can occur and in this case $\ell=n-2$.

The proof of part (3) is similar. □

Lemma 6.10. *Suppose* $n\ge 2$ *and let* $\mu=\mu^{(n-1)}$. *We have*

$$\Lambda_\mu{=}(n{-}1,...,n{-}1,0\,|\,0,1{-}m,...,1{-}m)=\mu^{(1,n-2)}+\mu^{(2,n-3)}+\cdots+\mu^{(n-1,0)}.\tag{6.8}$$

Consider the Kac module V_{Λ_μ}, $P(\Lambda_\mu)$ *has a subgraph* (*not necessarily closed*)

$$\mu^{(n-2)}\begin{array}{c}\rightarrowtail\lambda_1\rightarrowtail\\ \rightarrowtail\ 0\ \rightarrowtail\end{array}-\alpha_{\min},\tag{6.9}$$

where $\lambda_1=\mu^{(n-2)}-\alpha_{\min}$.

Proof. First, (6.8) follows from (A.4). Note that all weights in (6.9) are strongly primitive weights (they in fact correspond to unlinked codes defined in [21]). Since 0 is not a bottom primitive weight, there must be a weight linked to 0 in $U_{V_{\Lambda_\mu}}(0)$, and this weight has to be $-\alpha_{\min}$ by Lemma 6.3. Thus $0\rightarrowtail-\alpha_{\min}$. We see from (6.5) that $\mu^{(n-2)}$ is the only primitive weight of V_{Λ_μ} other than μ with non-zero 1-cohomology. By Lemma 6.3, we must have $\mu^{(n-2)}\rightarrowtail 0$ (since there must be some weight τ such that $\tau\overset{e}{\rightarrowtail}0$). Note that the primitive vector v_{λ_1} can be obtained from the primitive vector $v_{\mu^{(n-2)}}$, in fact, $v_{\lambda_1}=f_{\alpha_{\min}}v_{\mu^{(n-2)}}$ (indeed it is non-zero and strongly primitive). Thus we have

$\mu^{(n-2)} \twoheadrightarrowtail \lambda_1$. Obviously $\lambda_1 \twoheadrightarrowtail -\alpha_{\min}$ since $-\alpha_{\min}$ is the bottom primitive weight. If there exists some ν such that

$$\mu^{(n-2)} \rightarrowtail \nu \twoheadrightarrowtail \lambda_1 \quad \text{or} \quad \lambda_1 \twoheadrightarrowtail \nu \rightarrowtail -\alpha_{\min},$$

then since $\ell(\lambda_1) = \ell(\mu^{(n-2)}) - 1$, we must have $\ell(\nu) < \ell(\lambda_1)$. By Remark 6.2, we must have

$$\nu \in P(\Lambda_\mu) \cap P(\mu^{(n-2)}) \cap P(\lambda_1).$$

By Lemma 6.9, we have $\nu \in P_+(\mu^{(n-2)})$, but we see from (6.6) that such ν does not exist. Thus $\mu^{(n-2)} \rightarrowtail \lambda_1 \rightarrowtail -\alpha_{\min}$. □

Lemma 6.11. *In $V_{\mu^{(j)}}$, we have the following subgraph of $P(\mu^{(j)})$:*

$$\mu^{(j-1)} \begin{array}{c} \leftarrowtail \mu_-^{(j)} \leftarrowtail \mu^{(j)} \rightarrowtail \lambda_1 \rightarrowtail \\ \downarrow \\ \longleftarrow\!\!\!< \;\; 0 \;\; >\!\!\!\longrightarrow \end{array} - \alpha_{\min}\ , \tag{6.10}$$

with $\lambda_1 = \mu^{(j)} - \alpha_{\min}$, where if $j = 0$, the part $\mu^{(j-1)} \begin{array}{c}\scriptstyle \leftarrowtail \mu_-^{(j)} \leftarrowtail \\ \scriptstyle \longleftarrow\!< \end{array}$ is missing.

Proof. We prove by induction on j. First suppose $j = 0$. Then (6.10) is reduced to

$$\mu^{(0)} \begin{array}{c} \rightarrowtail \lambda_1 \rightarrowtail \\ \rightarrowtail 0 \rightarrowtail \end{array} - \alpha_{\min} \tag{6.11}$$

The part $\mu^{(0)} \rightarrowtail 0$ is clear since it is the dual of the part $0 \rightarrowtail \mu^{(0)}$ in $P(\Lambda_{\mu^{(0)}})$ which has been used to prove $H^1(\mathfrak{g}, L_{\mu^{(0)}}) \neq 0$. Since $-\alpha_{\min}$ is the only other primitive weight of $V_{\mu^{(j)}}$ with non-zero 1-cohomology and since 0 is not the bottom primitive weight, 0 must be linked to some primitive weight λ with arrow pointed to λ. By Lemma 6.3, we must have $0 \rightarrowtail -\alpha_{\min}$. The proof of the part $\mu^{(0)} \rightarrowtail \lambda_1 \rightarrowtail -\alpha_{\min}$ is similar to that of (6.9).

Next suppose $j > 0$. We observe the following facts:

Fact 1: As in case $j = 0$, we have $\mu^{(j)} \rightarrowtail 0$.

Fact 2: $\mu^{(j-1)} \leftarrowtail 0$: Note that $\mu^{(j-1)}, -\alpha_{\min}$ are the only primitive weights of $V_{\mu^{(j)}}$ beside $\mu^{(j)}$ which have non-zero 1-cohomology. We claim that 0 is not a strongly primitive weight of $V_{\mu^{(j)}}$. Otherwise, the bottom primitive weight τ of $V_{\Lambda_\mu^{(j)}}$ is in $P(V_0)$, but $\tau_1 = j > 0$ (note from [20, Theorem 3.2] that $\tau = \mu^{(j)} - \sum_{\alpha \in SW_{\mu^{(j)}}} \alpha$, where $SW\mu^{(j)}$ is the set of roots in the southwest chains of $\mu^{(j)}$, there is only one element of $SW\mu^{(j)}$ located on the first row, i.e., the n-th atypical root $\alpha_{\max}$), a contradiction (using [23, Proposition 3.5], one can also see that τ is not in $P(V_0)$). Thus there is a primitive weight λ in $U(0) = U_{V_{\Lambda_\mu}}(0)$ with level higher than 0. Thus λ has the form (5.10) satisfying (5.12). This λ must be $\mu^{(j-1)}$. Thus $\mu^{(j-1)} \twoheadleftarrowtail 0$ and $\mu^{(j-1)}$ is the highest weight in $\overline{M}(\mu^{(j-1)} \twoheadleftarrowtail 0)$. So we have a highest weight module $\overline{M}(\mu^{(j-1)} \twoheadrightarrowtail 0)$, which is a quotient of $V_{\mu^{(j-1)}}$. But by the inductive assumption, we have $\mu^{(j-1)} \rightarrowtail 0$ in $V_{\mu^{(j-1)}}$, thus also in $\overline{M}(\mu^{(j-1)} \twoheadrightarrowtail 0)$. Therefore, $\mu^{(j-1)} \leftarrowtail 0$ in $\overline{M}(\mu^{(j-1)} \twoheadleftarrowtail 0)$. Thus $\mu^{(j-1)} \leftarrowtail 0$.

Fact 3: $0 \rightarrowtail -\alpha_{\min}$: First note that $(-\alpha_{\min})^* = \mu^{(n-1)}$. As $\mu^{(j)} = \sum_{\alpha \in \Gamma} \alpha$ for a unique subset $\Gamma \subset \Delta_1^+$, there is up to scalars a unique $\mathfrak{g}_0$-highest weight vector of weight 0 which must be a primitive vector v_0 of weight 0. Similarly, there is up to scalars a unique $\mathfrak{g}_0$-highest weight vector v' of weight $-(\mu^{(n-1)})^R$ which must be the one corresponding to the lowest $\mathfrak{g}_0$-highest weight vector of $L_{-\alpha_{\min}}$, and we have

$$v' = \prod_{\alpha \in \Gamma_1} f_\alpha v_0, \tag{6.12}$$

where $\Gamma_1 = \{\alpha_{1,\dot{1}}, ..., \alpha_{m-1,\dot{1}}, \alpha_{m,\dot{1}}, \alpha_{m,\dot{2}}, ..., \alpha_{m,\dot{n}}\}$. Indeed the right hand side of (6.12) is non-zero (since v_0 has the leading term $\prod_{\alpha\in\Gamma} f_\alpha v_{\mu^{(j)}}$) and is a $\mathfrak{g}_0$-highest weight vector of weight $-(\mu^{(n-1)})^R$. This means $0 \mathrel{\rightarrowtail\!\!\!\twoheadrightarrow} -\alpha_{\min}$. We cannot have $\mu^{(j-1)} \mathrel{\rightarrowtail\!\!\!\twoheadrightarrow} -\alpha_{\min}$. Assume otherwise. Since $U(\mu^{(j-1)})$ is a quotient of $V_{\mu^{(j-1)}}$, by inductive assumption, we must have $\mu^{(j-1)} \rightarrowtail 0 \rightarrowtail -\alpha_{\min}$, contradicting $\mu^{(j-1)} \leftarrowtail 0$. Thus in $U(0)/U(\mu^{(j-1)})$ which is now a highest weight module, we still have $0 \mathrel{\rightarrowtail\!\!\!\twoheadrightarrow} -\alpha_{\min}$. So $0 \rightarrowtail -\alpha_{\min}$, since $-\alpha_{\min}$ is the unique primitive weight in $U(0)/U(\mu^{(j-1)})$ with non-zero 1-cohomology.

Fact 4: $\mu^{(j-1)} \leftarrowtail \mu_-^{(j)} \leftarrowtail \mu^{(j)}$: Note that $\mu_-^{(j)}, \mu^{(j-1)}$ are strongly primitive weights such that $\mu^{(j-1)} \mathrel{\twoheadleftarrow\!\!\!\leftarrowtail} \mu_-^{(j)} \mathrel{\twoheadleftarrow\!\!\!\leftarrowtail} \mu^{(j)}$ (in [21], the primitive vector $v_{\mu^{(j-1)}}$ is constructed from the primitive vector $v_{\mu_-^{(j)}}$). If $\mu^{(j-1)} \leftarrowtail \mu_-^{(j)} \leftarrowtail \mu^{(j)}$ is not valid, then there is some ν such that

$$\mu^{(j-1)} \leftarrowtail \nu \mathrel{\twoheadleftarrow\!\!\!\leftarrowtail} \mu_-^{(j)} \quad \text{or} \quad \mu_-^{(j)} \mathrel{\twoheadleftarrow\!\!\!\leftarrowtail} \nu \leftarrowtail \mu^{(j)}. \tag{6.13}$$

Such ν can only be in $P_+(\mu^{(j)}) \cap P(\mu_-^{(j)})$ by Lemma 6.8 and Remark 6.2. But by (6.7), we see that no ν can satisfy (6.13).

Fact 5: $\mu^{(j)} \rightarrowtail \lambda_1 \rightarrowtail -\alpha_{\min}$: The proof is similar to that of (6.9), (6.11) and Fact 4. □

Now consider dual graph of (6.10). Note that

$$(\mu_-^{(j)})^* = \mu_+^{(j-1)}, \quad \lambda_1^* = \mu^{(n-1)} + \mu^{(2,j)}.$$

In $P(\Lambda_{\mu^{(j)}})$, we have

$$\begin{array}{ccccc} & \longrightarrow & 0 & \longleftarrow & \\ \mu^{(j-1)} & & \downarrow & & (-\alpha_{\min})^* \\ & \to \mu_+^{(j-1)} \to & \mu^{(j)} & \leftarrow \lambda_1^* \leftarrow & \end{array} . \tag{6.14}$$

Lemma 6.12. *Denote $M_1 = M(\mu^{(j-1)} \rightarrowtail \mu_+^{(j-1)})$, $M_2 = M(\lambda_1^* \leftarrowtail (-\alpha_{\min})^*)$. We have $H^1(\mathfrak{g}, M_1) = H^1(\mathfrak{g}, M_2) = 0$.*

Proof. The arguments for the proof of $H^1(\mathfrak{g}, M_1) = 0$ are similar to those given after (6.18).

To prove $H^1(\mathfrak{g}, M_2) = 0$, first we consider the Kac module V_{λ_1}. Since we have a highest weight module with graph $\lambda_1 \rightarrowtail -\alpha_{\min}$ by (6.11), we must also have this in V_{λ_1}. It is straightforward to verify that $-\alpha_{\min}$ is the only primitive weight of V_{λ_1} with non-zero 1-cohomology and 0 is not a primitive weight of V_{λ_1}. Thus in the dual Kac module $V_{\lambda_1}^* = V_{2\rho_1-(\lambda_1)^R}$, we have $\lambda_1^* \leftarrowtail (-\alpha_{\min})^*$ (so M_2 is a submodule of $V_{\lambda_1}^*$) and $(-\alpha_{\min})^*$ is the only primitive weight with non-zero 1-cohomology while 0 is not a primitive weight. Thus $H^0(\mathfrak{g}, V_{\lambda_1}^*/M_2) = 0$. From the exact sequence $0 \to M_2 \to V_{\lambda_1}^* \to V_{\lambda_1}^*/M_2 \to 0$, we obtain

$$0 = H^0(\mathfrak{g}, V_{\lambda_1}^*/M_2) \to H^1(\mathfrak{g}, M_2) \to H^1(\mathfrak{g}, V_{\lambda_1}^*) = 0.$$

Thus we have the lemma. □

6.3. **Computation of second cohomology groups.** With the technical preparations in the last subsection, we can now compute the second cohomology groups with coefficients in the finite dimensional irreducible modules.

Suppose $H^2(\mathfrak{g}, L_\mu) \neq 0$. Then $\mu \neq 0$. Let $\varphi \in Z^2(\mathfrak{g}, L_\mu)^{\mathfrak{g}_0}$. As before, we can assume

$$\varphi|_{\mathfrak{g}_0^{ss}\times\mathfrak{g}} = 0 \quad \text{and} \quad \varphi|_{\check{\rho}_1\times\mathfrak{g}} \text{ is a 1-cocycle.} \tag{6.15}$$

6.3.1. *The case with $f_\alpha\varphi(e_\beta,e_\gamma)=0$ for $\beta,\gamma\in\Delta_1^+$ and $\varphi(e_\beta,e_\gamma)\neq 0$ for some $\beta,\gamma\in\Delta_1^+$.* Then $\varphi(e_\beta,e_\gamma)\in L_\mu^{bottom}$. This means that $\mu_{bottom}=2\alpha_{\max}$ or $\eta^{(1)}$ (in the latter case $n\geq 2$). We obtain (cf. [23, Proposition 3.5])

$$\mu=\mu_+^{(n-1)},\quad \text{ or }\quad \mu=\mu_-^{(n-1)} \text{ with } n\geq 2. \tag{6.16}$$

As in (4.9), we have $H^2(\mathfrak{g},L_\mu)=\mathbb{C}\overline{\phi_{2,1}}$.

6.3.2. *The case with $e_\alpha\varphi(f_\beta,f_\gamma)=0$ for $\beta,\gamma\in\Delta_1^+$ and $\varphi(f_\beta,f_\gamma)\neq 0$ for some $\beta,\gamma\in\Delta_1^+$.* In this case,

$$\mu=-2\alpha_{\min}=(\mu_+^{(n-1)})^*,\quad \text{ or }\quad \mu=\eta^{(2)}=(\mu_-^{(n-1)})^* \text{ with } n\geq 2. \tag{6.17}$$

and so we have $H^2(\mathfrak{g},L_\mu)\cong\mathbb{C}$.

6.3.3. *The case with $\mu=\mu^{(n-1)}$ or its dual $\mu=-\alpha_{\min}$.* It suffices to consider the case $\mu=-\alpha_{\min}$. First suppose $n=1$. Then $\Lambda_\mu=0$ and V_0 has two composition factors L_μ and $L_0=\mathbb{C}$. Since $H^1(\mathfrak{g},L_0)=H^2(\mathfrak{g},V_0)=0$, from $0\to L_\mu\to V_0\to L_0\to 0$, we have

$$0=H^1(\mathfrak{g},L_0)\to H^2(\mathfrak{g},L_\mu)\to H^2(\mathfrak{g},V_0)=0,$$

Thus we obtain $H^2(\mathfrak{g},L_\mu)=0$. So suppose $n\geq 2$. Then by (6.4) and (6.9), we obtain

$$\dim H^2(\mathfrak{g},L_\mu)=\dim H^1(\mathfrak{g},V_{\Lambda_\mu}/L_\mu)\leq \dim H^1(\mathfrak{g},M(\mu^{(n-2)}\rightarrowtail\mu_+^{(n-3)})).$$

We claim that $H^1(\mathfrak{g},M(\mu^{(n-2)}\rightarrowtail\mu_+^{(n-3)}))=0$. Otherwise, there exists a non-split extension of $M(\mu^{(n-2)}\rightarrowtail\mu_+^{(n-3)})$:

$$0\to M(\mu^{(n-2)}\rightarrowtail\mu_+^{(n-3)})\to\widehat{W}\to\mathbb{C}\to 0,$$

i.e., we have an indecomposable module with graph

$$0\rightarrowtail\mu^{(n-2)}\rightarrowtail\mu_+^{(n-3)}\quad \text{ or }\quad \mu^{(n-2)}\rightarrowtail\mu_+^{(n-3)}\leftarrowtail 0. \tag{6.18}$$

Since $H^1(\mathfrak{g},L_{\mu_+^{(n-3)}})=0$, the second case cannot occur. For the first case, we obtain that the lowest $\mathfrak{g}_0$-highest weight vector $v_\lambda\in L_{\mu_+^{(n-3)}}$ is of weight

$$\lambda=-(\mu_+^{(n-2)})^R=(0,-1,...,-1,-n\,|\,m,1,...,1,0)$$

(which is lower than the lowest $\mathfrak{g}_0$-highest weight of $L_{\mu^{(n-2)}}$) and can be generated by a primitive vector v_0 of weight 0 such that $v_\lambda\in U(\mathfrak{g}_{-1})v_0$. But $-\lambda$ cannot be written as a sum of distinct positive odd roots. This is a contradiction. Thus we have the claim. So we obtain

$$\dim H^2(\mathfrak{g},L_\mu)=0 \text{ if } \mu=\mu^{(n-1)} \text{ or } -\alpha_{\min}.$$

6.3.4. *The case with $\mu=\mu^{(j)}$ with $j\leq n-2$ (thus $n\geq 2$).* From (6.4) we obtain

$$\begin{aligned}\dim H^2(\mathfrak{g},L_{\mu^{(j)}})&=\dim H^1(\mathfrak{g},V_{\Lambda_{\mu^{(j)}}}/L_{\mu^{(j)}})\\ &\leq \dim H^1(\mathfrak{g},M_1)+\dim H^1(\mathfrak{g},M_2),\end{aligned}$$

where M_1, M_2 are as in Lemma 6.12. By Lemma 6.12, the far right hand side of the above inequality vanishes. Thus

$$\dim H^2(\mathfrak{g},L_{\mu^{(j)}})=0.$$

Before considering the next case with $\mu=\mu_\pm^{(\ell)}$, we first assume that μ is not any of the weights considered the earlier cases. Then $H^1(\mathfrak{g},V_\mu)=H^2(\mathfrak{g},V_\mu)=0$ by Theorems 3.1 and 4.1. Suppose $H^2(\mathfrak{g},L_\mu)\neq 0$. In (6.15), we can further suppose $\varphi(\check\rho_1,\mathfrak{g})=0$. We shall further suppose that $\varphi(e_\alpha,f_\beta)\neq 0$ for some α,β (otherwise it is already

considered above). Thus L_μ must contain a $\mathfrak{g}_0$-submodule isomorphic to an irreducible $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{+1}\wedge\mathfrak{g}_{-1}$, i.e., contain a $\mathfrak{g}_0$-highest weight which is one of the following

$$(1,0,...,0,-1\,|\,0,...,0),\ (0,...,0\,|\,1,0,...,0,-1),\ (1,0,...,0,-1\,|\,1,0,...,0,-1),\ 0. \tag{6.19}$$

Thus

$$\mu \text{ minus a weight in (6.19) is a sum of distinct positive odd roots.} \tag{6.20}$$

Since $H^2(\mathfrak{g},L_\mu)\neq 0$, there exists the exact sequence

$$0\to L_\mu\to V_1\to V_2\to \mathbb{C}\to 0. \tag{6.21}$$

This implies that V_1 has graph $\mu \leftarrowtail \tau$ for some τ, and V_2 has graph $\tau \leftarrowtail 0$. Thus

$$\tau=-\alpha_{\min},\ \mu^{(\ell)},\ 0\le \ell\le n-1 \quad\text{and}\quad \mu\in P(\tau)\cup P^\vee(\tau). \tag{6.22}$$

Therefore μ is a weight in (6.5)–(6.7).

First we give the duals of some useful weights:

$$(\mu_+^{(\ell)})^*=\begin{cases}\mu^{(n-2)}-\alpha_{\min} & \text{if}\quad \ell=n-2,\\ \mu_-^{(\ell+1)} & \text{if}\ \ 0\le\ell\le n-3,\end{cases} \tag{6.23}$$

and $(\mu_-^{(n-1)})^*=\eta^{(2)}$ (this weight has been considered in (6.16)), and

$$(\mu^{(\ell)}-\alpha_{\min})^*=\begin{cases}\mu_+^{(n-2)} & \text{if}\quad \ell=n-2,\\ \mu^{(n-1)}+\mu^{(2,\ell)} & \text{if}\ \ 0\le\ell\le n-3,\end{cases} \tag{6.24}$$

$$(\mu^{(\ell)}+\mu^{(2,j)})^*=\begin{cases}\mu^{(j)}-\alpha_{\min} & \text{if } \ell=n-1,\ j\le n-3,\\ \mu^{(\ell)}+\mu^{(2,j)} & \text{if } 0\le\ell\le n-2,\ 0\le j\le \ell-1,\ (\ell,j)\neq(n-2,n-3),\end{cases} \tag{6.25}$$

and

$$\begin{aligned}&(\mu^{(n-1)}+\mu^{(2,n-2)})^*=(0,...,0,-2,-2\,|\,2,2,0,...,0),\\ &(\mu^{(n-2)}+\mu^{(n-3)})^*=\mu_+^{(n-3)}+\eta^{(2)},\end{aligned}$$

(these two weights do not meet (6.20)).

6.3.5. *The case with $\mu=\mu_\pm^{(\ell)}$.*

Lemma 6.13. *If $\mu=\mu_+^{(\ell)}$, $0\le\ell\le n-2$, then* $\dim H^2(\mathfrak{g},L_\mu)=1$.

Proof. By (6.5) and (6.6), we see that $\mu^{(\ell)}$ is the only primitive weight of $V_{\mu_+^{(\ell)}}$ with non-zero 1-cohomology. In $V_{\mu_+^{(\ell)}}$, we have $\mu_+^{(\ell)} \rightarrowtail \mu^{(\ell)}$ since by (6.14) and by Remark 6.2, we see that a highest weight module with graph $\mu_+^{(\ell)} \rightarrowtail \mu^{(\ell)}$ exists (thus in $V_{\mu_+^{(\ell)}}$, it must be so). By considering the dual Kac module $(V_{\mu_+^{(\ell)}})^*$, it is straightforward to see that $\dim H^1(\mathfrak{g},(V_{\mu_+^{(\ell)}})^*/L^*_{\mu_+^{(\ell)}})=1$. Thus $\dim H^2(\mathfrak{g},L_{\mu_+^{(\ell)}})=\dim H^2(\mathfrak{g},L^*_{\mu_+^{(\ell)}})=1$. $\square$

6.3.6. *The case with* $\mu=\mu^{(\ell)}+\mu^{(2,\ell-1)}$, $1\le\ell\le n-3$.

Lemma 6.14. *If* $\mu=\mu^{(\ell)}+\mu^{(2,\ell-1)}$, $1\le\ell\le n-3$, *then* $H^2(\mathfrak{g},L_\mu)=0$.

Proof. First we prove that there is a subgraph of $P(\mu)$,

$$\begin{array}{ccccc} & & \mu & & \\ & & \downarrow & & \\ & \leftarrowtail\ \lambda_1 \leftarrowtail & \mu^{(\ell)} & \rightarrowtail \lambda_2 \rightarrowtail & \\ \mu^{(\ell-2)} & & \downarrow & & -\,\alpha_{\min}\ , \\ & \longleftarrowtail & 0 & \rightarrowtail\!\longrightarrow & \end{array} \tag{6.26}$$

where $\lambda_1=\mu^{(\ell)}+\mu^{(2,\ell-2)}$, $\lambda_2=\mu^{(\ell)}-\alpha_{\min}$, and where if $\ell=1$, the part $\mu^{(\ell-2)}\overset{\leftarrowtail\lambda_1\leftarrowtail}{\longleftarrowtail}$ is missing.

We assume that $\ell\ge2$ as the case $\ell=1$ can be regarded as a special case (cf. the proof of Lemma 6.11). We observe the following Facts:

Fact 0′: One can check that all weights in (6.26) are primitive weights of $P(\mu)$ and all primitive weights of $P(\mu)$ with non-zero 1-cohomology appear in (6.26).

Fact 1′: $\mu^{(\ell)}\twoheadrightarrowtail\lambda_1$: Since $\mu^{(\ell)}$ is not a strongly primitive weight (as in Fact 2 of Lemma 6.11), $\mu^{(\ell)}\twoheadrightarrowtail\lambda$ for some $\lambda\in P(\mu)\cap P^\vee(\mu^{(\ell)})$ by Remark 6.2. By using (6.5), one can check that $P(\mu)\cap P^\vee(\mu^{(\ell)})=\{\lambda_1\}$. (From what to be discussed below, we can see that we must have $\mu^{(\ell)}\rightarrowtail\lambda_1$, since if $\mu^{(\ell)}\rightarrowtail\nu\twoheadrightarrowtail\lambda_1$ for some ν, then $\nu\in P_+(\mu^{(\ell)})\cap P(\lambda_1)\cap P(\mu)$, but there is no such ν.)

Fact 2′: We do not have $\lambda_1\twoheadrightarrowtail 0$: Otherwise, λ_1 is a strongly primitive weight (corresponding to an unlinked code), thus $U(\lambda_1)$ is a quotient of V_{λ_1}. From the dual Kac module $V^*_{\lambda_1}$, one can see (using the same arguments as that given in the paragraph after (5.30)) that in V_{λ_1}, we have $\lambda_1\twoheadrightarrowtail\mu^{(\ell)}\twoheadrightarrowtail 0$, and so we must also have this in $U(\lambda_1)$. This contradicts the fact that $\mu^{(\ell)}\twoheadrightarrowtail\lambda_1$.

Fact 3′: $\mu^{(\ell)}\rightarrowtail 0$: First we have $\mu^{(\ell)}\twoheadrightarrowtail 0$ (as the proof before (5.31)). Then consider $U(\mu^{(\ell)})/U(\lambda_1)$ which is now a quotient of $V_{\mu^{(\ell)}}$ (since there is no primitive weight in $U(\mu^{(\ell)})/U(\lambda_1)$ with level higher than that of $\mu^{(\ell)}$ by Fact 1′), we see that $\mu^{(\ell)}\rightarrowtail 0$ (since in $V_{\mu^{(\ell)}}$ we have this).

Fact 4′: $0\overset{e}{\rightarrowtail}\mu^{(\ell-2)}$: This follows from the same arguments in Fact 2 of Lemma 6.11.

Fact 5′: $\lambda_1\to\mu^{(\ell-2)}$: We must have $\lambda_1\twoheadrightarrowtail\mu^{(\ell-2)}$, otherwise by Fact 1′, we would have $\mu^{(\ell-2)}\in P(U(\mu^{(\ell)})/U(\lambda_1))\subset P(\mu^{(\ell)})$, but $P(\mu^{(\ell)})$ does not contain $\mu^{(\ell-2)}$ by (6.5). If not $\lambda_1\rightarrowtail\mu^{(\ell-2)}$, then $\lambda_1\twoheadrightarrowtail\lambda\rightarrowtail\mu^{(\ell-2)}$ for some λ of $P(\mu)\cap P(\lambda_1)$ (since λ_1 is strongly primitive). By Lemma 6.8 (if $\lambda\overset{f}{\rightarrowtail}\mu^{(\ell-2)}$) or Lemma 6.9(1) and (2) (if $\lambda\overset{e}{\rightarrowtail}\mu^{(\ell-2)}$), we see that such λ does not exist.

Fact 6′: As in the proof of (6.10), we do not have $\mu^{(\ell-2)}\twoheadrightarrowtail-\alpha_{\min}$, but $0\rightarrowtail-\alpha_{\min}$ is valid. Consider the highest weight module $U(\mu^{(\ell)})/U(\lambda_1)$, we have the part $\mu^{(\ell)}\rightarrowtail 0\rightarrowtail-\alpha_{\min}$ of the graph, and by (6.10), we must also have the part $\mu^{(\ell)}\rightarrowtail\lambda_2\rightarrowtail-\alpha_{\min}$.

Fact 7′: We have proved (6.26) except the part $\mu\rightarrowtail\mu^{(\ell)}$. But this must be valid. Otherwise, there is another primitive weight λ with $\lambda\rightarrowtail\mu^{(\ell)}$, thus we cannot have $\lambda\overset{f}{\rightarrowtail}\mu^{(\ell)}$ since λ_1 is the only weight of higher level linked to $\mu^{(\ell)}$ (Fact 1′). We cannot have $\lambda\overset{e}{\rightarrowtail}\mu^{(\ell)}$ either, since by Lemmas 6.8 and 6.9, there is no other weight of lower level linked to $\mu^{(\ell)}$. Thus we have proved (6.26).

Now consider the dual Kac module V_μ^*. By Lemma 6.6, we have

$$\begin{aligned}\dim H^2(\mathfrak{g}, L_\mu) &= \dim H^2(\mathfrak{g}, L_\mu^*) = \dim H^1(\mathfrak{g}, V_\mu^*/L_\mu^*)\\ &\le \dim H^1(\mathfrak{g}, M(\mu^{(\ell-2)} \rightarrowtail \lambda_1^*))\\ &\quad + \dim H^1(\mathfrak{g}, M(0 \rightarrowtail \mu^{(\ell)})) + \dim H^1(\mathfrak{g}, M((-\alpha_{\min})^* \rightarrowtail \lambda_2^*))\\ &= 0,\end{aligned}$$

where the last equality follows from the same arguments in the proof of Lemma 6.12. □

6.3.7. *The case with* $\mu = \mu^{(\ell)} + \mu^{(2,j)}$, $0 \le j \le \ell - 2 \le n - 3$.

Lemma 6.15. *If* $\mu = \mu^{(\ell)} + \mu^{(2,j)}$, $0 \le j \le \ell - 2 \le n - 3$, *then* $\dim H^2(\mathfrak{g}, L_\mu) = 1$.

Proof. We prove by induction on $\ell + j$. We shall suppose $j \ge 1$ and $\ell \le n - 2$ since the case $j = 0$ or $\ell = n - 1$ will become a special case in the following discussion. We need to prove the existence of the following subgraphs of $P(\mu)$,

$$\begin{gathered}\lambda_3 \rightarrowtail -\alpha_{\min} \leftarrowtail 0 \leftarrowtail \mu^{(\ell)} \leftarrowtail \mu \rightarrowtail \mu^{(j)},\\ \lambda_1 \rightarrowtail \mu^{(j-1)},\ \lambda_2 \rightarrowtail \mu^{(\ell-1)},\end{gathered} \tag{6.27}$$

where $\lambda_1 = \mu^{(\ell)} + \mu^{(2,j-1)}$, $\lambda_2 = \mu^{(\ell-1)} + \mu^{(2,j)}$, $\lambda_3 = \mu^{(\ell)} - \alpha_{\min}$ and $\lambda_4 = \mu^{(\ell-1)} + \mu^{(2,j-1)}$. To do this, we need to observe the following facts:

Fact 0″: All weights appeared in the graphs are primitive weights of V_μ and all primitive weights with non-zero 1-cohomology appear in the above graphs.

Fact 1″: (a) Since $\mu^{(\ell)}$ cannot be strongly primitive, we have $\mu^{(\ell)} \twoheadrightarrowtail \lambda_1$ as in Fact 1′ of Lemma 6.14, and $P(\mu) \cap P^\vee(\mu^{(\ell)}) = \{\lambda_1, \mu, \mu^{(\ell)}\}$. (b) By considering the highest weight modules $U(\lambda_1)$ and $U(\mu^{(\ell)})/U(\lambda_1)$, we see that every primitive weight derived from $\mu^{(\ell)}$ is in $P(\mu^{(\ell)}) \cup P(\lambda_1)$.

Fact 2″: We have $\mu^{(\ell)} \twoheadrightarrowtail 0 \twoheadrightarrowtail -\alpha_{\min}$ as in the proof of Fact 3 and Fact 6′ of previous lemmas. Thus we must have $\mu^{(\ell)} \begin{smallmatrix}\rightarrowtail \lambda_3 \rightarrowtail\\ \rightarrowtail 0 \rightarrowtail\end{smallmatrix} -\alpha_{\min}$.

Fact 3″: Similar to Fact 1″(a), we have $\mu^{(j)} \twoheadrightarrowtail \lambda_2$ since $P(\mu) \cap P^\vee(\mu^{(j)}) = \{\lambda_2, \mu, \mu^{(j)}\}$.

Fact 4″: (a) We have $\mu \rightarrowtail \mu^{(\ell)}$. Otherwise suppose $\mu \twoheadrightarrowtail \nu \rightarrowtail \mu^{(\ell)}$, then by Fact 1″, $\nu \in P(\mu) \cap P_+(\mu^{(\ell)})$. Take

$$\Lambda = \mu^{(\ell)} + \mu^{(2,j)} + \mu^{(3,j-1)} + \cdots + \mu^{(j+1,0)}.$$

One can check that μ and $\mu^{(\ell)}$ are strongly primitive weight of $P(\Lambda)$, and $\mu \twoheadrightarrowtail \mu^{(\ell)}$ in $P(\Lambda)$. In fact in V_Λ, the $\mathfrak{g}_0$-highest weight for each of them is unique, and one can easily construct the corresponding $\mathfrak{g}_0$-highest weight vector of $L_{\mu^{(\ell)}}$ from that of L_μ (cf. arguments in the paragraph of (5.31)). Define a homomorphism ϕ from $V_\mu \to V_\Lambda$ by sending v_μ to the corresponding primitive vector in V_Λ. Note that any element of $P_+(\mu^{(\ell)})$ is either a strongly primitive weight in $P(\Lambda)$ or not a primitive weight in $P(\Lambda)$. Thus in $P(\Lambda)$ we do not have $\nu \overset{e}{\rightarrowtail} \mu^{(\ell)}$. So in $P(\mu)$ we do not have this either. Thus $\mu \rightarrowtail \mu^{(\ell)}$.

(b) We have $\mu \rightarrowtail \mu^{(j)}$. Otherwise, there exists some $\nu \in P_+(\mu^{(j)})$ such that $\mu \twoheadrightarrowtail \nu \rightarrowtail \mu^{(j)}$. Note that none of the elements of $P_+(\mu^{(j)}) \backslash \{\mu^{(j)}\}$ is a primitive weight of $P(\Lambda)$ (thus the homomorphism ϕ defined in (a) maps a primitive vector v_ν in V_μ of weight ν to zero). The map ϕ sends a primitive vector v_{λ_2} in V_μ of weight λ_2 to the corresponding primitive vector v'_{λ_2} of the same weight λ_2 in V_Λ. This can be seen from the following arguments: Note that λ_2 is a strongly primitive weight in $P(\Lambda)$

such that its primitive vector can be constructed from that of μ in V_Λ. This is because λ_2 is 2 levels lower than μ, and in V_Λ the atypical roots of Λ corresponding to μ (i.e., atypical roots corresponding to non-zero columns of the code for Λ which defines μ, cf. [6, 21]) are all disconnected from (in sense of [23]) or not c-related to (cf. Definition A.2) the other 2 atypical roots corresponding to λ_2. Therefore, by Fact 3$''$, ϕ must map a primitive vector $v_{\mu^{(j)}}$ in V_μ of weight $\mu^{(j)}$ to a primitive vector $v'_{\mu^{(j)}}$ in V_Λ which can generate v'_{λ_2}. In particular, $\phi(v_{\mu^{(j)}})\neq 0$. But $\nu \rightarrowtail \mu^{(j)}$ means that $\phi(v_{\mu^{(j)}})$ is generated by $\phi(v_\nu)=0$. This contradiction shows that we cannot have $\mu \rightarrowtail\!\!\!\twoheadrightarrow \nu \rightarrowtail \mu^{(j)}$.

Fact 5$''$: We have $\lambda_2 \rightarrowtail\!\!\!\twoheadrightarrow \lambda_4$, since both are strongly primitive weights and λ_4 is two levels lower than λ_2, one can construct v_{λ_4} from v_{λ_2}.

Fact 6$''$: Applying Facts 5$''$, 1$''$(a) and 4$''$(a) to the case $\ell-1$ shows that $\lambda_2 \rightarrowtail \mu^{(\ell-1)} \rightarrowtail\!\!\!\twoheadrightarrow \lambda_4$. Similarly, $\lambda_1 \rightarrowtail\!\!\!\twoheadrightarrow \lambda_4$, from this and Facts 3$''$ and 4$''$(b), we have $\lambda_1 \rightarrowtail \mu^{(j-1)}$. This proves (6.27).

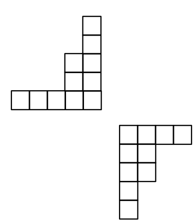
Now in the dual Kac module V_{Λ_μ}, take quotient V_{Λ_μ}/L_μ and denote $P_1 = M(0 \rightarrowtail \mu^{(\ell)}))$, $P_2 = L_{\mu^{(j)}}$, $P_3 = M(\mu^{(j-1)} \rightarrowtail \lambda_1^*))$, $P_4 = M(\mu^{(\ell-1)} \rightarrowtail \lambda_2^*))$, $P_5 = M((-\alpha_{\min})^* \rightarrowtail \lambda_3^*))$, from $H^2(\mathfrak{g},L_\mu)=H^1(\mathfrak{g},V_{\Lambda_\mu}/L_\mu)$, by Lemmas 6.6 and 6.7, we obtain

$$1=\dim H^1(\mathfrak{g},P_2)\leq \dim H^2(\mathfrak{g},L_\mu)\leq \sum_{i=1}^{5}\dim H^1(\mathfrak{g},P_i)=1,$$

where the last equality follows from the same arguments in the proof of Lemma 6.12. $\square$

6.3.8. *The remaining case.* For a weight μ, we denote by $\mathcal{K}(\mu)$ and $\mathcal{L}(\mu)$ the multiplicities of $\mathfrak{g}_0$-highest weight $2\alpha_{\max}$ in V_μ and in L_μ respectively. We need to compute $\mathcal{K}(\mu)$ and $\mathcal{L}(\mu)$ for some weights μ. To do this, we use a composite Young Diagram to denote a weight, e.g., if $\lambda=(\lambda^1\,|-\lambda^2)=(4,2,2,1,1\,|-1,-1,-3,-5)$, then the composite Young Diagram of λ is

$(\lambda^2)^*\ =$



$=\ \lambda^1$.

In general, for a weight $\lambda=(\lambda^1\,|-\lambda^2)=(\lambda_1,...,\lambda_m\,|-\lambda_{\dot 1},...,-\lambda_{\dot n})$ (in the rest of this section, we use this notation to denote a weight which is a little different from (2.2)), all $\lambda_i\geq 0$, we place the Young Diagram of λ^1 on the right side which consists of boxes in m rows such that the i-th row has λ_i's boxes for $i=1,...,m$, and we place the opposite Young Diagram $(\lambda^2)^*$ on the top which consists of boxes in n columns such that the ν-th column has $\lambda_{\dot\nu}$'s boxes for $\nu=1,...,n$.

Since V_μ is completely reducible as a $\mathfrak{g}_0$-module, we have

$$\begin{aligned}\mathcal{K}(\lambda)&=\dim \mathrm{Hom}_{\mathfrak{g}_0}(L^{(0)}_{2\alpha_{\max}},U(\mathfrak{g}_{-1})\otimes L^{(0)}_\lambda)=\dim\mathrm{Hom}_{\mathfrak{g}_0}(U(\mathfrak{g}_{+1})\otimes L^{(0)}_{2\alpha_{\max}},L^{(0)}_\lambda)\\ &=\ \text{the multiplicity of } \mathfrak{g}_0\text{-highest weight } \lambda \text{ in } U(\mathfrak{g}_{+1})\otimes L^{(0)}_{2\alpha_{\max}}.\end{aligned}$$

Note that a $\mathfrak{g}_0$-highest weight λ of $U(\mathfrak{g}_{+1})\otimes L^{(0)}_{2\alpha_{\max}}$ can always be obtained as the sum of a $\mathfrak{g}_0$-highest weight μ of $U(\mathfrak{g}_{+1})$ and a weight ν of $L^{(0)}_{2\alpha_{\max}}$ under the following rule: Since a weight ν of $L^{(0)}_{2\alpha_{\max}}$ has the form

$$\nu=(0,...,0,\overset{i}{1},0,...,0,\overset{j}{1},0,...,0\,|\,0,...,\overset{p}{-1},0,...,0,\overset{q}{-1},0,...,0)$$

(where if $i=j$ the i-th coordinate becomes 2 and if $p=q$ the p-th coordinate becomes -2), when adding ν to μ, we require that $\mu_i\neq\mu_j$ and $\mu_p\neq\mu_q$ and that the resulting weight $\mu+\nu$ be dominant. This amounts to adding 2 boxes to the Young Diagram μ^1 such that they are not placed in the same column, and adding 2 boxes to the opposite Young Diagram $(\mu^2)^*$ such that they are not placed in the same row. Equivalently, μ is obtained from λ by removing 2 boxes of λ^1 from different columns and removing 2 boxes of $(\lambda^2)^*$ from different rows.

Since we only need to calculate $\mathcal{K}(\lambda)$ for λ being a $\mathfrak{g}_0$-highest weight of $U(\mathfrak{g}_{+1})$, λ (and also μ) must has the form (5.10) satisfying (5.12), thus we can only remove 2 boxes from different rows and different columns of λ^1 (and removing the corresponding boxes from $(\lambda^2)^*$). Thus for λ of the form (5.10) satisfying (5.12),

$$\begin{array}{rcl}\mathcal{K}(\lambda) & = & \text{the number of ways to remove 2 boxes of } \lambda^1 \text{ from}\\ & & \text{different rows and different columns such that}\\ & & \text{the remaining Young Diagram is still standard.}\end{array}\tag{6.28}$$

From this, it is straightforward to compute

$$\mathcal{K}(\mu^{(\ell)})=\begin{cases}0 & \text{if } \ \ell=0,\\ 1 & \text{if } \ 1\le\ell\le n-1.\end{cases}\tag{6.29}$$

Note from (6.7) that the only possible primitive weights λ of $\mu^{(\ell)}$ with $\mathcal{L}(\lambda)\neq0$ are the first 3 weights in (6.7). This and (6.29) already provide sufficient information to obtain by induction on ℓ,

$$\mathcal{L}(\mu^{(\ell)}_+)=1,\quad \mathcal{L}(\mu^{(\ell)})=\mathcal{L}(\mu^{(\ell)}_-)=0.\tag{6.30}$$

Similarly, we can compute

$$\mathcal{K}(\mu^{(\ell)}+\mu^{(2,j)})=\begin{cases}0 & \text{if } \ j=\ell-1=0,\\ 1 & \text{if } \ 1\le j=\ell-1\le n-2.\\ 3 & \text{if } \ j=0,\, 2\le\ell\le n-1,\\ 6 & \text{if } \ 1\le j\le\ell-2\le n-3.\end{cases}$$

Lemma 6.16. *Let $\mu=\mu^{(\ell)}+\mu^{(2,j)}$, $0\le j\le\ell-2\le n-3$, we have $\mathcal{L}(\mu)=1$.*

Proof. By (6.27), we have a module V with graph $\mu^{(j)}\rightarrowtail\mu$. Let $v'_{\alpha_{\max}}$ be the $\mathfrak{g}_0$-highest weight vector of weight $\alpha_{\max}$ in $L_{\mu^{(j)}}$ (cf. Remark 5.7), and let v_μ be the highest weight vector in L_μ. Then we must have $v_\mu=\prod_{\alpha\in\Gamma}e_\alpha v'_{\alpha_{\max}}$ for some subset $\Gamma\subset\Delta_1^+$. If $\mathcal{L}(\mu)=0$, then $e_{\alpha_{\max}}v'_{\alpha_{\max}}=0$, so $\alpha_{\max}\notin\Gamma$. Thus Γ is a unique subset such that $\mu=\sum_{\alpha\in\Gamma\cup\{\alpha_{\max}\}}\alpha$ (note that the way of writing μ as a sum of distinct positive odd roots is unique by observing from (2.3) that each $\mu^{(i,j)}$ corresponds to a unique subset S_{ij} of Δ_1^+ such that $\mu^{(i,j)}=\sum_{\alpha\in S_{ij}}\alpha$). Then we obtain a non-zero vector $v'_{\mu^{(\ell)}}=\prod_{\alpha\in\Gamma'}e_\alpha v'_{\alpha_{\max}}\in V$, where $\Gamma'=\{\alpha_{1,\dot n-\ell},...,\alpha_{1,\dot n-1},\alpha_{2,\dot n},...,\alpha_{m-n+\ell+1,\dot n}\}$, that is a $\mathfrak{g}_0$-highest weight vector of weight $\mu^{(\ell)}$. The weight $\mu^{(\ell)}$ cannot be a $\mathfrak{g}_0$-highest weight of $L_{\mu^{(j)}}$ since its level is higher than that of $\mu^{(j)}$ (as $\ell>j$), it cannot be a $\mathfrak{g}_0$-highest weight of L_μ either since L_μ does not contain a $\mathfrak{g}_0$-highest weight $\mu^{(\ell)}$ (it is straightforward to see that in the Kac module V_μ, there is up to scalars a unique $\mathfrak{g}_0$-highest weight vector of weight $\mu^{(\ell)}$, which is in $L_{\mu^{(\ell)}}$). This is a contradiction against the fact that V has graph $\mu^{(j)}\rightarrowtail\mu$. $\square$

Lemma 6.17. *Let*

$$\mu = \mu^{(1,j_1)} + \cdots + \mu^{(k,j_k)},\ n-1 \geq j_1 > j_2 > \ldots > j_k \geq 0. \tag{6.31}$$

Then for any $\lambda \in P(\mu)$, $\mathcal{L}(\lambda) = 1$ *if and only if*

$$\lambda = \mu_+^{(i)}, \quad \mu^{(\ell)} + \mu^{(2,j)}, \quad 0 \leq i \leq n-1,\ 0 \leq j \leq \ell - 2 \leq n-3. \tag{6.32}$$

Proof. Case (1): First suppose $j_p \geq j_{p+1} + 3$ for $p = 1, \ldots, k-1$ and $j_k \geq 1$. It is straightforward to calculate

$$\mathcal{K}(\mu) = (2k-1) + (2k-2) + \cdots + 1 = k(2k-1). \tag{6.33}$$

We can check that the following $k(2k-1)$ weights are primitive weights of V_μ,

$$\begin{aligned} &\mu_+^{(j_p-1)}, \quad \mu^{(j_p)} + \mu^{(2,j_q)}, \quad \mu^{(j_p)} + \mu^{(2,j_q-1)}, \\ &\mu^{(j_p-1)} + \mu^{(2,j_q)}, \quad \mu^{(j_p-1)} + \mu^{(2,j_q-1)}, \quad 1 \leq p, q \leq k, \ \ p < q, \end{aligned} \tag{6.34}$$

which have non-zero multiplicities of $\mathfrak{g}_0$-highest weight $2\alpha_{\max}$ by Lemma 6.16 and (6.30).

Case 2: Suppose there are t number of j_i's such that $j_i = j_{i+1} + 2$, say the indices are $r_1, \ldots, r_t$. Then we still have (6.33). However, the t weights $\mu^{(j_{r_p}-1)} + \mu^{(2,j_{r_p+1})}$, $p = 1, \ldots, t$, in the list (6.34) are no longer primitive, but on the other hand, we have t other primitive weights $\mu_+^{(j_{r_p}-2)}$, $p = 1, \ldots, t$, which should be added to the list.

Case 3: Suppose further there are t' number of j_i's such that $j_i = j_{i+1} + 1$, say the indices are $r'_1, \ldots, r'_{t'}$. Then when using (6.28) to compute $\mathcal{K}(\mu)$, the removals from rows or columns in $\mu^{(r'_p, j_{r'_p})}$, $p = 1, \ldots, t'$, should not be counted. Thus $\mathcal{K}(\mu) = (k-t')(2(k-t')-1)$. Similarly, all weights in (6.34) with indices $j_{r'_p}$, $p = 1, \ldots, t'$, should be removed from the list, thus the total number is still $\mathcal{K}(\mu)$.

Case 4: Suppose $j_k = 0$. Then $\mathcal{K}(\mu) = \sum_{i=1}^{2k-2} i = (k-1)(2k-1)$. In this case, there are $1 + 2(k-1)$ weights $\mu_+^{(j_k-1)}$, $\mu^{(j_p)} + \mu^{(2,j_k-1)}$, $\mu^{(j_p-1)} + \mu^{(2,j_k-1)}$ in (6.34) which should not in the list, the total number is again $\mathcal{K}(\mu)$. $\square$

Now we prove the main result on the second cohomology groups with coefficients in finite dimensional irreducible modules.

Theorem 6.18. *Let* L_μ *be the finite-dimensional irreducible* $\mathfrak{g}$*-module with highest weight* μ*. Then* $\dim H^2(\mathfrak{g}, L_\mu) \leq 1$*. Furthermore,* $H^2(\mathfrak{g}, L_\mu) \neq 0$ *if and only if* μ *is one of the following weights*

$$\mu_+^{(\ell)}, \quad \mu_-^{(j)}, \quad \mu^{(p)} + \mu^{(2,q)}, \quad \mu^{(j-1)} - \alpha_{\min}, \quad -2\alpha_{\min}, \quad \eta^{(2)}, \tag{6.35}$$

where $0 \leq \ell \leq n-1$, $\ 1 \leq j \leq n-1$, $\ 0 \leq p \leq q-2 \leq n-3$, *and in the case* $n = 1$, $\eta^{(2)}$ *will not appear. The total number of weights in* (*6.35*) *is* $\frac{(n+1)(n+2)}{2} - \delta_{n,1}$.

Proof. Let μ be a weight in (6.5)–(6.7) (recall the statement after (6.22) that in order for an irreducible module to have non-trivial second cohomology, its highest weight must be in (6.5)–(6.7)), but we assume that μ is not a weight in (6.35), nor a weight already considered in the previous cases. Being a weight in (6.5)–(6.7), μ is a primitive weight in the Kac module with a highest weight of the form (6.31), thus $\mathcal{L}(\mu) = 0$. Furthermore, μ has the form

$$\mu = \sum_{i=1}^{k} \mu^{(i,j_i)} + \theta_1 \alpha - \theta_2 \alpha_{\min}, \tag{6.36}$$

where $\theta_1, \theta_2 \in \{0, 1\}$ such that at most one of θ_1, θ_2 is non-zero (cf. the statement after (6.5)), and α is some atypical root. Moreover $k \geq 2$, and $k \geq 3$ if $\theta_1 = \theta_2 = 0$.

We shall show that L_μ has trivial second cohomology by contradiction. In order for L_μ to have non-zero second cohomology, we either have $\mu^{(j)} \rightarrowtail \mu$ for $j \leq n-1$, or $-\alpha_{\min} \rightarrowtail \mu$. We only need to consider the former case, as the latter case is the dual situation for $j = n-1$.

Assume $\mu^{(j)} \rightarrowtail \mu$. Then we have the following module: $V := M(\mu^{(j)} \rightarrowtail \mu)$. By (6.22), $\mu \in P^\vee(\mu^{(j)}) \cup P(\mu^{(j)})$. Since at least one coordinate of $\mu^{(j)} - \mu$ is ≤ 0, we have $\mu \in P^\vee(\mu^{(j)})$. So $j \leq j_1$ and $\theta_2 = 0$ by (6.5). Suppose $v_\mu = \prod_{\beta \in \Gamma} e_\beta v'_{\alpha_{\max}}$ for some subset $\Gamma \subset \Delta_1^+$ such that $\alpha_{\max} \notin \Gamma$, where v_μ, $v'_{\alpha_{\max}}$ are as in Lemma 6.16. This means that $\mu = \sum_{\beta \in \Gamma \cup \{\alpha_{\max}\}} \beta$ is a sum of distinct positive odd roots. In this case, Γ must contain a subset Γ' such that $\lambda := \mu^{(j_1)} + \mu^{(2,j_2)} = \sum_{\beta \in \Gamma' \cup \{\alpha_{\max}\}} \beta$, and $v'_\lambda = \prod_{\beta \in \Gamma'} e_\beta v'_{\alpha_{\max}} \neq 0$ is a $\mathfrak{g}_0$-highest weight vector of weight λ in V. However, the weight λ cannot be a $\mathfrak{g}_0$-highest weight of $L_{\mu^{(j)}}$ (since $\mu^{(j)} - \lambda$ has negative coordinates) or L_μ (since the Kac module V_μ has up to scalars a unique $\mathfrak{g}_0$-highest weight vector of weight λ which appears in the composition factor L_λ of V_μ), thus a contradiction results.

Therefore, there does not exist any weight ν with non-zero 1-cohomology such that $\nu \rightarrowtail \mu$. This in turn implies that $H^2(\mathfrak{g}, L_\mu) = 0$. $\square$

Remark 6.19. *The dual weight μ^* of any weight μ in (6.35) is given by equations (6.17), (6.23)–(6.25). From these equations we can see that μ^* also belongs to the list (6.35).*

Remark 6.20. *Suppose μ is a weight such that all the central elements of $U(\mathfrak{g})$ contained in $\mathfrak{g}U(\mathfrak{g})$ act trivially on L_μ. Then as in Remark 5.7, by Theorem 6.18, one can prove that $H^2(\mathfrak{g}, L_\mu) \neq 0$ if and only if L_μ contains a copy of irreducible $\mathfrak{g}_0$-module $L_{2\alpha_{\max}}$, $L_{-2\alpha_{\min}}$, $L_{\eta^{(1)}}$ or $L_{\eta^{(2)}}$, that is, L_μ contains a copy of irreducible $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{+1} \wedge \mathfrak{g}_{+1}$ or $\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1}$.*

7. Cohomology groups with coefficients in enveloping algebra

We continue to denote the special linear superalgebra $\mathfrak{sl}_{m|n}$ by $\mathfrak{g}$. The aim of this section is to prove the following

Theorem 7.1. *Regarding the universal enveloping algebra $U(\mathfrak{g})$ as a $\mathfrak{g}$-module under the adjoint action, we have $H^1(\mathfrak{g}, U(\mathfrak{g})) \neq 0$, and $H^2(\mathfrak{g}, U(\mathfrak{g})) = 0$.*

The fact that $H^1(\mathfrak{g}, U(\mathfrak{g})) \neq 0$ follows from Corollary 3.2. Also, in the case $n = 1$, it has been proved in [17] that $H^2(\mathfrak{g}, U(\mathfrak{g})) = 0$.

To prove the remaining part of the theorem, we need some preparations. Recall that $U(\mathfrak{g})$, regarded as a $\mathfrak{g}$-module, is canonically isomorphic to the supersymmetric algebra $S(\mathfrak{g})$ (see [15]). Also, $S(\mathfrak{g})$ is a direct summand in the $\mathfrak{g}$-module $S(\mathfrak{gl}(m|n))$. If we let $\mathbb{C}^{m|n}$ denote the natural $\mathfrak{gl}(m|n)$-module, and $\bar{\mathbb{C}}^{m|n}$ its dual, then $\mathfrak{gl}(m|n) \cong \mathbb{C}^{m|n} \otimes \bar{\mathbb{C}}^{m|n}$. Here $\mathfrak{gl}(m|n)$ acts on $\mathbb{C}^{m|n} \otimes \bar{\mathbb{C}}^{m|n}$ as the diagonal subalgebra of $\mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n)$, where the latter superalgebra acts on $\mathbb{C}^{m|n} \otimes \bar{\mathbb{C}}^{m|n}$ in the obvious way. The $\mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n)$ action extends uniquely to $S(\mathbb{C}^{m|n} \otimes \bar{\mathbb{C}}^{m|n})$. Now $S(\mathbb{C}^{m|n} \otimes \bar{\mathbb{C}}^{m|n})$ is isomorphic to the subalgebra [18, Definition 3.3] of regular functions on the general linear supergroup. A Peter-Weyl type theorem [18, Proposition 3.1]

states that as a $\mathfrak{gl}(m|n)\times\mathfrak{gl}(m|n)$-module,

$$S(\mathbb{C}^{m|n}\otimes\bar{\mathbb{C}}^{m|n})=\bigoplus_{\lambda}L_\lambda\otimes L_\lambda^*,\tag{7.1}$$

where λ runs over all weights $\lambda=(\lambda_1,...,\lambda_m\,|\,\lambda_{\dot 1},...,\lambda_{\dot n})$ such that

$$\lambda_1\ge\lambda_2\ge...\ge\lambda_m\ge0,\ \ \lambda_{\dot 1}\ge\lambda_{\dot 2}\ge...\ge\lambda_{\dot n}\ge0,\ \ \lambda_m\ge\#\{\nu\in\mathbf{I}_2\,|\,\lambda_\nu\ne0\}\tag{7.2}$$

(thus in this section, λ will not satisfy the condition in (2.2)).

Consider the action of the diagonal $\mathfrak{gl}(m|n)$ subalgebra of $\mathfrak{gl}(m|n)\times\mathfrak{gl}(m|n)$ on $S(\mathbb{C}^{m|n}\otimes\bar{\mathbb{C}}^{m|n})$, and restrict it to an $\mathfrak{g}$-action. Then $S(\mathfrak{g})$ regarded as a $\mathfrak{g}$-module under the adjoint action can be embedded as a direct summand in $S(\mathbb{C}^{m|n}\otimes\bar{\mathbb{C}}^{m|n})$.

The proof of the theorem thus shifts its focus to the $\mathfrak{g}$-submodules $L_\lambda\otimes L_\lambda^*$ of $S(\mathbb{C}^{m|n}\otimes\bar{\mathbb{C}}^{m|n})$. We divide the proof into a series of technical lemmas.

Lemma 7.2. *If $V=V_\mu\otimes W$, where μ is an integral dominant weight and W is any finite-dimensional $\mathfrak{g}$-module, then there exists a filtration (called a Kac flag)*

$$0=V_0\subset V_1\subset\cdots\subset V_k=V,\tag{7.3}$$

such that V_i/V_{i-1} is a Kac module for $i=1,...,k$ (we call such a Kac module a factor Kac module of V).

Proof. Let $\mathfrak{g}_{\ge0}=\mathfrak{g}_0+\mathfrak{g}_{+1}$, and denote by $L^{(0)}_\mu$ the irreducible $\mathfrak{g}_{\ge0}$-module with highest weight μ. Then $V=\big(U(\mathfrak{g})\otimes_{U(\mathfrak{g}_{\mathfrak{g}\ge0})}L^{(0)}_\mu\big)\otimes W=U(\mathfrak{g})\otimes_{U(\mathfrak{g}_{\mathfrak{g}\ge0})}(L^{(0)}_\mu\otimes W)$, where $L^{(0)}_\mu\otimes W$ is regarded as a $\mathfrak{g}_{\ge0}$-module with the obvious action. Since the induction functor $U(\mathfrak{g})\otimes_{U(\mathfrak{g}_{\mathfrak{g}\ge0})}-$ is exact, by applying it to any composition series of the finite dimensional $\mathfrak{g}_{\ge0}$-module $L^{(0)}_\mu\otimes W$, we produce a Kac flag for V. □

Lemma 7.3. *$H^2(\mathfrak{g},L_\lambda\otimes L_\lambda^*)=0$ if λ satisfying (7.2) is typical (i.e., $(\lambda+\rho,\alpha)\ne0$ for all $\alpha\in\Delta_1^+$).*

Proof. If λ is typical then $L_\lambda^*=V_{\lambda^*}$ is a Kac module (in this case $\lambda^*=2\rho_1-\lambda^R$), thus we can use Lemma 7.2. Note that the highest weight vector of each factor Kac module of $L_\lambda\otimes L_\lambda^*$ has level $\le\ell(\lambda)+\ell(\lambda^*)=\ell(2\rho_1)$. Thus $H^2(\mathfrak{g},V_i/V_{i-1})=0$ by Theorem 4.1. Then from the short exact sequence $0\to V_{i-1}\to V_i\to V_i/V_{i-1}\to0$, by (2.11), we obtain $H^2(\mathfrak{g},V_i)=0$ by induction on i. □

Lemma 7.4. *$H^2(\mathfrak{g},L_\lambda\otimes L_\lambda^*)=0$ if λ satisfying (7.2) is atypical.*

Proof. Recall that the (i,ν)-entry $A(\lambda)_{i,\nu}$ of the atypicality matrix $A(\lambda)$ of λ is (cf. (A.1))

$$A(\lambda)_{i,\nu}=(\lambda+\rho,\alpha_{i,\dot\nu})=\lambda_i+\lambda_{\dot\nu}+m-i+1-\nu\ \text{ for }\ i=1,...,m,\ \nu=1,...,n,\tag{7.4}$$

(the smallest element is $A(\lambda)_{m,\dot n}=\lambda_m+\lambda_{\dot n}+1-n$), from this we see that λ is atypical only if $\lambda_m\le n$ (cf. (7.2)). One observes that the northeast chains NE_λ of λ (cf. [6], see also Examples A.4 and A.5) satisfy

$$\begin{aligned}&(m,n)\in NE_\lambda,\\&(i,j)\in NE_\lambda\implies(k,\ell)\in NE_\lambda\ \text{ if }\ k\ge i\text{ and }\ \ell\ge j.\end{aligned}\tag{7.5}$$

We have the short exact sequence

$$0\to L_\lambda\otimes L_\lambda^*\to V_{\Lambda_\lambda}\otimes L_\lambda^*\to(V_{\Lambda_\lambda}/L_\lambda)\otimes L_\lambda^*\to0.$$

Since $\mathbb{C}$ is a submodule of $L_\lambda \otimes L_\lambda^*$ (also of $V_{\Lambda_\lambda} \otimes L_\lambda^*$), we also have another short exact sequence

$$0 \to (L_\lambda \otimes L_\lambda^*)/\mathbb{C} \to (V_{\Lambda_\lambda} \otimes L_\lambda^*)/\mathbb{C} \xrightarrow{f} (V_{\Lambda_\lambda}/L_\lambda) \otimes L_\lambda^* \to 0.$$

Thus by (2.11),

$$\begin{array}{rcl} H^1(\mathfrak{g}, (V_{\Lambda_\lambda} \otimes L_\lambda^*)/\mathbb{C}) & \xrightarrow{f^1} & H^1(\mathfrak{g}, (V_{\Lambda_\lambda}/L_\lambda) \otimes L_\lambda^*) \\ \to \quad H^2(\mathfrak{g}, (L_\lambda \otimes L_\lambda^*)/\mathbb{C}) & \to & H^2(\mathfrak{g}, (V_{\Lambda_\lambda} \otimes L_\lambda^*)/\mathbb{C}). \end{array} \tag{7.6}$$

Note from Lemma 6.5 that $H^2(\mathfrak{g}, (V_{\Lambda_\lambda} \otimes L_\lambda^*)/\mathbb{C}) = H^2(\mathfrak{g}, V_{\Lambda_\lambda} \otimes L_\lambda^*) = 0$, where the last equality follows from the proof of Lemma 7.3 (note that in this case $\lambda^* = 2\rho_1 - (\Lambda_\lambda)^R$ since $-\lambda^*$ is the lowest weight of V_{Λ_λ}). Thus

$$H^2(\mathfrak{g}, L_\lambda \otimes L_\lambda^*) = H^2(\mathfrak{g}, (L_\lambda \otimes L_\lambda^*)/\mathbb{C}) \cong H^1(\mathfrak{g}, (V_{\Lambda_\lambda}/L_\lambda) \otimes L_\lambda^*)/\mathrm{Im}(f^1). \tag{7.7}$$

Claim 1. $\dim H^1(\mathfrak{g}, (V_{\Lambda_\lambda}/L_\lambda) \otimes L_\lambda^*) = 1$.

To prove this claim, we first examine the short exact sequence

$$0 \to (V_{\Lambda_\lambda}/L_\lambda) \otimes L_\lambda^* \to (V_{\Lambda_\lambda}/L_\lambda) \otimes V_{\Lambda_{\lambda^*}} \to (V_{\Lambda_\lambda}/L_\lambda) \otimes (V_{\Lambda_{\lambda^*}}/L_{\lambda^*}) \to 0,$$

which leads to the exact sequence

$$H^0(\mathfrak{g}, (V_{\Lambda_\lambda}/L_\lambda) \otimes (V_{\Lambda_{\lambda^*}}/L_{\lambda^*})) \to H^1(\mathfrak{g}, (V_{\Lambda_\lambda}/L_\lambda) \otimes L_\lambda^*) \to H^1(\mathfrak{g}, (V_{\Lambda_\lambda}/L_\lambda) \otimes V_{\Lambda_{\lambda^*}}). \tag{7.8}$$

Subclaim a) $\dim H^1(\mathfrak{g}, (V_{\Lambda_\lambda}/L_\lambda) \otimes V_{\Lambda_{\lambda^*}}) \le 1$.

To prove this subclaim, by Lemma 7.2 and Theorem 3.1, it suffices to prove that

$$V_{2\rho_1} \text{ is not a factor Kac module of } W = (V_{\Lambda_\lambda}/L_\lambda) \otimes V_{\Lambda_{\lambda^*}}, \tag{7.9}$$

$$V_{2\rho_1 + \alpha_{\max}} \text{ is a factor Kac module of } W \text{ with multiplicity} \le 1. \tag{7.10}$$

Since $W = U(\mathfrak{g}_{-1})((V_{\Lambda_\lambda}/L_\lambda) \otimes L^{(0)}_{\Lambda_{\lambda^*}})$, suppose $w_{2\rho_1} \in (V_{\Lambda_\lambda}/L_\lambda) \otimes L^{(0)}_{\Lambda_{\lambda^*}}$ is a primitive vector of weight $2\rho_1$, then $w_{2\rho_1}$ is a combination of the form

$$u \otimes v, \;\; u \in V_{\Lambda_\lambda}/L_\lambda, \, v \in L^{(0)}_{\Lambda_{\lambda^*}}, \tag{7.11}$$

and u is in some $\mathfrak{g}_0$-highest weight submodule $L^{(0)}_\eta$ of $V_{\Lambda_\lambda}/L_\lambda$ (for some weight η such that $2\rho_1 = \eta' + \Lambda_{\lambda^*}$, where η' is a weight in $L^{(0)}_\eta$) and the level of η is

$$\ell(\eta) = \ell(2\rho_1) - \ell(\Lambda_{\lambda^*}). \tag{7.12}$$

Note that with restriction to $\mathfrak{g}_0$, we have $(2\rho_1)|_{\mathfrak{g}_0} = 0$, thus the fact $2\rho_1 = \eta' + \Lambda_{\lambda^*}$ means that

$$\eta|_{\mathfrak{g}_0} = \big(\Lambda_{\lambda^*}|_{\mathfrak{g}_0}\big)^o, \tag{7.13}$$

where we use λ^o to denote the dual of a weight λ with respect to $\mathfrak{g}_0$ (i.e., in fact $\lambda^o|_{\mathfrak{g}_0} = -\lambda^R|_{\mathfrak{g}_0}$). But we have

$$\big(\Lambda_{\lambda^*}|_{\mathfrak{g}_0}\big)^o = -(\Lambda_{\lambda^*})^R|_{\mathfrak{g}_0} = -(2\rho_1 - \lambda)|_{\mathfrak{g}_0} = \lambda|_{\mathfrak{g}_0}. \tag{7.14}$$

Equations (7.12)–(7.14) imply that $\eta = \lambda$. Recall that (cf. (A.4))

$$\Lambda_\lambda = \lambda + \sum_{(i,\nu) \in NE_\lambda} \alpha_{i,\dot{\nu}}. \tag{7.15}$$

By (7.5), we see that when writing $\Lambda_\lambda - \lambda$ as a sum of distinct positive odd roots, there is only one way. This shows that in V_{Λ_λ}, there is only one copy of $\mathfrak{g}_0$-highest weight λ which occurs in L_λ. This contradicts that $\eta = \lambda$ is a $\mathfrak{g}_0$-highest weight of $V_{\Lambda_\lambda}/L_\lambda$. This proves (7.9).

Next suppose $w_{2\rho_1+\alpha_{\max}} = \sum u \otimes v$ (as in (7.11)) is a primitive vector of weight $2\rho_1 + \alpha_{\max}$. As discussion above, now η must be $\lambda + \alpha$ for some positive odd roots $a = \alpha_{i,\dot\nu}$. Suppose $\alpha_{i,\dot\nu} \notin NE_\lambda$. Then we must have some position of the i-th row and some position of the ν-th column in NE_λ (say if none of positions of the i-th row is in NE_λ, then the i-th coordinate of $\lambda + \alpha_{i,\dot\nu}$ is bigger than that of Λ_λ by (7.15), a contradiction). Say $(i, \nu_1), (i_1, \nu) \in NE_\lambda$, then by (7.5), we have

$$\Lambda_\lambda - (\lambda + \alpha_{i,\dot\nu}) = \sum_{(k,\ell)\in NE_\lambda\setminus\{(i_1,\nu),(i,\nu_1)\}} \alpha_{k,\dot\ell} + \alpha_{i_1,\dot\nu_1},$$

(where $\alpha_{i_1,\dot\nu_1}$ appears twice), which cannot be written as a sum of distinct positive odd roots. Thus we must have $\alpha_{i,\dot\nu} \in NE_\lambda$. Note from (7.4) and (7.5) that all atypical roots of λ are c-related in sense of [21] (cf. Definition A.2) or connected in sense of [23], $\lambda + \alpha_{i,\dot\nu}$ is dominant only if (i, ν) is located at the rightmost and the topmost position of NE_λ, such position is unique by (7.5). This proves (7.10) and Subclaim a).

Subclaim b) $H^0(\mathfrak{g}, (V_{\Lambda_\lambda}/L_\lambda) \otimes (V_{\Lambda_{\lambda^*}}/L_{\lambda^*})) = 0.$

We have

$$\begin{aligned} H^0(\mathfrak{g}, (V_{\Lambda_\lambda}/L_\lambda) \otimes (V_{\Lambda_{\lambda^*}}/L_{\lambda^*})) &= ((V_{\Lambda_\lambda}/L_\lambda) \otimes (V_{\Lambda_{\lambda^*}}/L_{\lambda^*}))^{\mathfrak{g}} \\ &= \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, (V_{\Lambda_\lambda}/L_\lambda) \otimes (V_{\Lambda_{\lambda^*}}/L_{\lambda^*})) \\ &= \mathrm{Hom}_{\mathfrak{g}}((V_{\Lambda_{\lambda^*}}/L_{\lambda^*})^*, V_{\Lambda_\lambda}/L_\lambda) = \mathrm{Hom}_{\mathfrak{g}}(M_\lambda, V_{\Lambda_\lambda}/L_\lambda), \end{aligned}$$

where M_λ is the maximal proper submodule of V_λ (note that $(V_{\Lambda_{\lambda^*}})^* = V_\lambda$ since the lowest weight of $V_{\Lambda_{\lambda^*}}$ is $(\Lambda_{\lambda^*} - 2\rho_1)^R = -\lambda$). Suppose μ is a primitive weight of M_λ, then $\mu = \lambda - \sum_{\alpha\in S} \alpha$, where S is some subset of Δ_1^+ such that at least one atypical root γ of λ is in S, and all roots in S are $\le \gamma$ (see [6]; this fact can also be proved by [23, Conjecture 4.1] and [1, Main Theorem]). By (7.15), we have $\Lambda_\lambda - \mu = \sum_{\alpha\in S\cup NE_\lambda} \alpha$. From the property (7.5) of NE_λ and the fact that $\gamma \in S \cap NE_\lambda$, one can easily see that $\sum_{\alpha\in S\cup NE_\lambda} \alpha$ cannot be written as a distinct sum of positive odd roots. This means that μ is not a primitive weight of V_{Λ_λ}, that is, M_λ and V_{Λ_λ} do not have a common primitive weight, which implies that $\mathrm{Hom}_{\mathfrak{g}}(M_\lambda, V_{\Lambda_\lambda}/L_\lambda) = 0$.

Subclaim c) $\dim H^1(\mathfrak{g}, (V_{\Lambda_\lambda}/L_\lambda) \otimes L_\lambda^*) \ge 1$ (this together with (7.8) and Subclaims a) and b) proves Claim 1).

Note that the top $\mathfrak{g}_0$-highest weight submodule of $V_{\Lambda_\lambda}/L_\lambda$ is $\mathfrak{g}_0$-dual of $L^{(0)}_{\lambda^*}$. There exists a $\mathfrak{g}_0$-highest weight vector

$$v_{2\rho_1} = v_{\Lambda_\lambda} \otimes w + \cdots \quad (w \text{ is the lowest weight vector of } L^{(0)}_{\lambda^*})$$

of weight $2\rho_1$. This vector must be strongly primitive since there is no vector with level higher than that of $2\rho_1$. Let $W = U(\mathfrak{g})w_{2\rho_1}$. Take

$$v = \prod_{i>1,\,\nu<n} f_{\alpha_{i,\dot\nu}} v_{2\rho_1} = \Big(\prod_{i>1,\,\nu<n} f_{\alpha_{i,\dot\nu}} v_{\Lambda_\lambda}\Big) \otimes w + \cdots . \tag{7.16}$$

Note that in V_{Λ_λ}, the primitive vector v_λ of L_λ has the form

$$v_\lambda = \prod_{\alpha\in NE_\lambda} f_\alpha v_{\Lambda_\lambda} + \cdots .$$

By (7.5), we see that $NE_\lambda \not\subset \{(i,\nu) \,|\, i > 1,\ \nu < n\}$, thus

$$\prod_{i>1,\,\nu<n} f_{\alpha_{i,\dot\nu}} v_{\Lambda_\lambda} \ne 0 \quad (\text{as a vector in } V_{\Lambda_\lambda}/L_\lambda).$$

Thus the first term of (7.16) is non-zero, clearly this term cannot be cancelled by any other terms in (7.16), i.e., $v \neq 0$.

Note that W is a quotient module of the Kac module $V_{2\rho_1}$, and in $V_{2\rho_1}$ the primitive vector of weight $\mu^{(n-1)} = (n, 1, ..., 1 \,|\, -1, ..., -1, -m)$ is precisely defined by (7.16) (cf. [21, Theorem 6.6]). Thus v is a primitive vector of weight $\mu^{(n-1)}$. We claim that $U(\mathfrak{g})v$ is an irreducible submodule $L_{\mu^{(n-1)}}$ of W: First W does not contain a trivial submodule since $(V_{\Lambda_\lambda}/L_\lambda) \otimes L^*_\lambda$ does not contain a trivial submodule as the proof of Subclaim b), and in $V_{2\rho_1}$, $\mu^{(n-1)}$ is the only primitive weight links to primitive weight 0 since $\mu^{(n-1)}$ is the only primitive weight of $V_{2\rho_1}$ with non-zero 1-cohomology. Thus $U(\mathfrak{g})v = L_{\mu^{(n-1)}}$ is an irreducible submodule of W (and of $(V_{\Lambda_\lambda}/L_\lambda) \otimes L^*_\lambda$), this shows that $H^1(\mathfrak{g}, (V_{\Lambda_\lambda}/L_\lambda) \otimes L^*_\lambda) \neq 0$ by Lemma 6.7. This proves Subclaim c) and Claim 1.

Claim 2. In (7.6), the map f^1 is surjective, thus by (7.7), $H^2(\mathfrak{g}, L_\lambda \otimes L^*_\lambda) = 0$.

Similar to the proof of Subclaim c), $(V_{\Lambda_\lambda} \otimes L^*_\lambda)/\mathbb{C}$ has an irreducible submodule $V_1 = L_{\mu^{(n-1)}}$ which maps under f to the irreducible submodule $V_2 = L_{\mu^{(n-1)}}$ of $(V_{\Lambda_\lambda}/L_\lambda) \otimes L^*_\lambda$, thus induces the map $f^1 : H^1(\mathfrak{g}, V_1)$ onto $H^1(\mathfrak{g}, V_2)$ by (2.11). Hence f^1 is onto. This proves Lemma 7.3. □

Finally we return to the proof of Theorem 7.1. By (7.1) and Lemmas 7.2 and 7.3, we have

$$H^2(\mathfrak{g}, U(\mathfrak{g})) = H^2(\mathfrak{g}, S(\mathfrak{g})) \subset \bigoplus_{\lambda} H^2(\mathfrak{g}, L_\lambda \otimes L^*_\lambda) = 0.$$

This completes the proof.

8. Cohomology groups of the Lie superalgebra $C(n)$

In this last section, we generalize the results of previous sections to the other type I classical Lie superalgebra $C(n) = \mathfrak{osp}_{2|2n-2}$, which is a $\mathbb{Z}$-graded subalgebra of $\mathfrak{sl}_{2|2n-2}$ (see for example [22]) such that $C(n) = C(n)_{-1} \oplus C(n)_0 \oplus C(n)_{+1}$ with

$$C(n)_0 = \left\{ \left(\begin{array}{cc|cc} \alpha & -\alpha & & \\ \hline & & \beta & \gamma \\ & & \delta & -\beta^T \end{array} \right) \in \mathfrak{sl}_{2|2n-2} \;\middle|\; \alpha \in \mathbb{C},\ \beta, \gamma, \delta \in \mathfrak{gl}_{n-1},\ \gamma^T = \gamma,\ \delta^T = \delta \right\}$$
$$\cong \mathbb{C} \oplus \mathfrak{sp}_{2n-2},$$

$$C(n)_{-1} = \left\{ \left(\begin{array}{c|cc} & 0 & 0 \\ & \xi & \eta \\ \hline -\eta^T & 0 & \\ \xi^T & 0 & \end{array} \right) \in \mathfrak{sl}_{2|2n-2} \;\middle|\; \xi, \eta \text{ are row vectors of dimension } n-1 \right\},$$

$$C(n)_{+1} = \left\{ \left(\begin{array}{c|cc} & \xi & \eta \\ & 0 & 0 \\ \hline 0 & -\eta^T & \\ 0 & \xi^T & \end{array} \right) \in \mathfrak{sl}_{2|2n-2} \;\middle|\; \xi, \eta \text{ are row vectors of dimension } n-1 \right\},$$

where the up-index "T" stands for the transpose of a matrix or vector.

Denote $\epsilon = \epsilon_1$, $\delta_i = \epsilon_{i+2}\, i = 1, 2, ..., n-1$. Then we have

$$\Delta_0^+ = \{\delta_i - \delta_j,\ \delta_i + \delta_k \,|\, 1 \le i, j, k \le n-1,\ i < j\}, \quad \Delta_1^+ = \{\epsilon \pm \delta_j \,|\, 1 \le j \le n-1\}.$$

Denote

$$\begin{aligned} h_1 &= E_{11} - E_{22} + E_{33} - E_{n+2,n+2}, \\ h_i &= E_{i+1,i+1} - E_{i+2,i+2} - E_{n+i,n+i} + E_{n+i+1,n+i+1}\ (2 \le i \le n-1), \\ h_n &= E_{n+1,n+1} - E_{2n,2n}, \end{aligned}$$

which forms a basis of the Cartan subalgebra $\mathfrak{h}$. A weight $\lambda = \lambda_0 \epsilon + \sum_{i=1}^{n-1} \lambda_i \delta_i \in \mathfrak{h}^*$

can be written as

$$\lambda=(\lambda_0|\lambda_1,...,\lambda_{n-1})=[a_1;a_2,...,a_n],\quad \text{where } a_i=\lambda(h_i),$$

and $a_i=\lambda_{i-1}+\lambda_i\,(1\le i\le n-1)$, $a_n=\lambda_{n-1}$. We have

$$\rho_0=\sum_{i=1}^{n-1}(n-i)\delta_i,\quad \rho_1=(n-1)\epsilon,$$

and $\alpha_{\min}=\epsilon-\delta_1$, $\alpha_{\max}=\epsilon+\delta_1$ are respectively the minimal and maximal positive odd roots.

Theorem 8.1. *We have*

$$\dim H^1(C(n),V_\lambda)=\begin{cases}1 & \text{if } \lambda=2\rho_1,2\rho_1+\alpha_{\max},\\ 0 & \text{otherwise},\end{cases}\tag{8.1}$$

$$\dim H^2(C(n),V_\lambda)=\begin{cases}1 & \text{if } \lambda=2\rho_1+\alpha_{\max},2\rho_1+2\alpha_{\max},\\ 0 & \text{otherwise},\end{cases}\tag{8.2}$$

$$\dim H^1(C(n),L_\lambda)=\begin{cases}1 & \text{if } \lambda=-\alpha_{\min},2\rho_1,\\ 0 & \text{otherwise},\end{cases}\tag{8.3}$$

$$\dim H^2(C(n),L_\lambda)=\begin{cases}1 & \text{if } \lambda=2\rho_1+\alpha_{\max},-2\alpha_{\min},\\ 0 & \text{otherwise},\end{cases}\tag{8.4}$$

and

$$H^1(C(n),U(C(n)))\ne 0\quad\text{and}\quad H^2(C(n),\mathrm{U}(C(n)))=0.\tag{8.5}$$

Proof. Note that the Kac module V_λ over $C(n)$ is at most singly atypical (for some discussions of Kac module over $C(n)$, see for example [22]). The proofs of (8.1) and (8.2) are exactly similar to those of Theorem 3.1 and Theorem 4.1 for the special case $\mathfrak{sl}_{m|1}$.

Suppose $H^1(C(n),L_\lambda)\ne 0$, then we have $\lambda\leftarrowtail 0$ (recall Definition 6.1). If λ has level lower than 0, then $\lambda\leftarrowtail 0$ is a highest weight module of highest weight 0 and thus is the Kac module V_0. In this case λ is the only other primitive weight of V_0, i.e., $\lambda=-\alpha_{\min}$. If λ has level higher than 0, then we have a module $\lambda\rightarrowtail 0$, which is now the Kac module V_λ such that 0 is a primitive weight, thus $\lambda=2\rho_1$. Hence,

$$H^1(C(n),L_\lambda)\ne 0\implies \lambda=-\alpha_{\min},\,2\rho_1,$$

from this one immediate obtains (8.3) using (2.11).

Now suppose $H^2(C(n),L_\lambda)\ne 0$, then we have $\lambda\leftarrowtail\mu$ for some μ with $H^1(C(n),L_\mu)$. Since either $\lambda\leftarrowtail\mu$ or $\lambda\rightarrowtail\mu$ must be a Kac module, from this we obtain $\lambda=2\rho_1+\alpha_{\max},\,-2\alpha_{\min}$. From this one immediate obtains (8.4) using (2.11).

The proof of the first equation of (8.5) is again similar as for the case of the special linear superalgebra. As for the second equation, note that regarded as a $C(n)$-module under the adjoint action, $U(C(n))$ does not have any weight with level higher than that of $2\rho_1$, thus $L_{2\rho_1+\alpha_{\max}}$ is not a composition factor of $U(C(n))$. Similarly, $L_{-2\alpha_{\min}}$ cannot appear as a composition factor of $U(C(n))$ either, since the lowest $C(n)_{\bar 0}$-highest weight of $L_{-2\alpha_{\min}}$ is $-2\rho_1-\alpha_{\min}$, which is not a weight of $U(C(n))$. Thus no composition factors of $U(C(n))$ has non-zero 2-cohomology, which implies $H^2(C(n),U(C(n)))=0$. □

Appendix A. Atypicality

We briefly recall the definitions [6, 21] of atypical roots $\gamma_1,..,\gamma_r$, atypicality matrix $A(\mu)$, *nqc*-relationship of atypical roots, and northeast chains NE_μ, of any weight μ,

and also illustrate the concepts by some examples. We consider the special linear superalgebra $\mathfrak{g} = \mathfrak{sl}_{m|n}$ only. Results presented here are used in Section 5.

Definition A.1. *We define the atypicality matrix of μ to be the $m \times n$ matrix $A(\mu) = (A(\mu)_{i,\eta})_{m\times n}$, where*

$$A(\mu)_{i,\eta} = (\mu + \rho, \alpha_{i,\dot\eta}) = \mu_i + \mu_{\dot\eta} + m - i + 1 - \eta,\ 1 \le i \le m,\ 1 \le \eta \le n. \quad \text{(A.1)}$$

An odd root $\alpha_{i,\dot\eta}$ is an atypical root of μ if $A(\mu)_{i,\eta} = 0$. We label the atypical roots of an r-fold atypical dominant integral weight μ by $\gamma_1 < ... < \gamma_r$.

Definition A.2. (1) *For $1 \le s,\, t \le r$, let x_{st} be the entry in $A(\mu)$ at the intersection of the column containing the γ_s zero with the row containing the γ_t zero, then $x_{st} \in \mathbb{Z}_+\backslash\{0\}$ for $s < t$ and $x_{ts} = -x_{st}$. Denote h_{st} the hook length between the zeros corresponding to γ_s, γ_t, i.e., the number of steps to go from the γ_s zero via x_{st} to the γ_t zero with the zeros themselves included in the count.*

(i)	*γ_s, γ_t are normally related (n)*	$\Longleftrightarrow$	$x_{st} > h_{st} - 1$;
(ii)	*γ_s, γ_t are quasi-critically related (q)*	$\Longleftrightarrow$	$x_{st} = h_{st} - 1$;
(iii)	*γ_s, γ_t are critically related (c)*	$\Longleftrightarrow$	$x_{st} < h_{st} - 1$.

It is straightforward to show that the q-relation is transitive, i.e., if γ_s, γ_t are q-related and γ_t, γ_u are q-related, then γ_s, γ_u are q-related.

(2) *The nqc-type (atypicality type) of an r-fold atypical μ is a triangular array*

$$nqc(\mu) = \begin{array}{llll} d_{1r} \cdots d_{sr} \cdots d_{tr} \cdots 0 \\ \vdots \qquad \vdots \qquad \vdots \quad \cdot^{\cdot^{\cdot}} \\ d_{1t} \cdots d_{st} \cdots 0 \\ \vdots \qquad \vdots \quad \cdot^{\cdot^{\cdot}} \\ d_{1s} \cdots 0 \\ \vdots \quad \cdot^{\cdot^{\cdot}} \\ 0 \end{array}$$

where the zeros correspond to $\{\gamma_1, \cdots, \gamma_r\}$ and $d_{st} = n, q, c \iff \gamma_s, \gamma_t$ are n-, q-, c-related respectively.

Definition A.3. *Denote by $D = \{(i, \eta)\,|\, 1 \le i \le m,\ 1 \le \eta \le n\}$ the set of positions of $A(\mu)$. For $1 \le s \le r$, let (b_s, c_s) be the position corresponding to γ_s. The east chain $E_\mu(s)$ emanating from (b_s, c_s) is a sequence of positions in D, whose entry are < 0 except the entry of (b_s, c_s), starting at (b_s, c_s) and extending in an easternly or north-easternly direction until it reaches the last column or the position defferent from (b_s, c_s) whose entry is ≥ 0, or it cannot extend further without leaving $A(\mu)$ by passing above its first row. For all η with $c_s \le \eta \le n$, $E_\mu(s)$ has at most one element in the η-th column; if $c_s \le \eta < n$, the row of the position in the $(\eta + 1)$-th column is $a_{\dot\eta}$ rows above the row of the position in the η-th column, where $a_{\dot\eta}$ is a Dynkin label of μ; if this is not possible, i.e., if this row would be the M-th row where $M < 1$, or the entry of this position is ≥ 0, then $E_\mu(s)$ ends in the η-th column, i.e., has no position to the right of the η-th column. Thus $E_\mu(s)$ is the maximal set of positions $(b_s, c_s) = (i_0, \eta_0), (i_1, \eta_1), ..., (i_k, \eta_k)$ satisfying*

$$\eta_{p+1} = \eta_p + 1 \le n,\ i_{p+1} = i_p - a_{\dot\eta_p} \ge 1,\ A(\mu)_{i_{p+1},\eta_{p+1}} < 0, \quad p = 0, 1, ..., k-1. \quad \text{(A.2)}$$

Similarly, the nouth chain $N_\mu(s)$ emanating from (b_s, c_s) is the maximal set of positions $(b_s, c_s) = (i_0, \eta_0), (i_1, \eta_1), ..., (i_k, \eta_k)$ satisfying

$$i_{p+1} = i_p - 1 \ge 1,\ \eta_{p+1} = \eta_p + a_{i_{p+1}} \le n,\ A(\mu)_{i_{p+1},\eta_{p+1}} > 0, \quad p = 0, 1, ..., k-1. \quad \text{(A.3)}$$

Set $NE_\mu(s) = N_\mu(s) \cup E_\mu(s)$, *and* $NE_\mu = \cup_{s=1}^r NE_\mu(s)$, *called the set of northeast chains of* μ.

We shall illustrate the northeast chains by giving two examples below. First note from [20, Theorem 3.2] or [23, Conjecture 4.1] that

$$\Lambda_\mu = \mu + \sum_{\alpha \in NE_\mu} \alpha. \tag{A.4}$$

Example A.4. $\mu = (3,2,2,1,0,0 \,|\, 0,0,-1,-3,-4) = [1,0,1,1,0;0;0,1,2,1]$.

$$A(\mu) = \begin{array}{c} 1\\0\\1\\1\\0 \end{array} \begin{array}{c} \begin{pmatrix} 8&7&5&2&0\\ 6&5&3&0&\bar2\\ 5&4&2&\bar1&\bar3\\ 3&2&0&\bar3&\bar5\\ 1&0&\bar2&\bar5&\bar7\\ 0&\bar1&\bar3&\bar6&\bar8 \end{pmatrix} \\ 0\ \ 1\ \ 2\ \ 1 \end{array}, \quad NE_\mu = \begin{array}{c} 1\\0\\1\\1\\0 \end{array} \begin{array}{c} \begin{pmatrix} *&*&*&1&5\\ *&*&1&4&1\\ *&*&1&1&*\\ *&1&3&*&*\\ 1&2&1&*&*\\ 1&1&*&*&* \end{pmatrix} \\ 0\ \ 1\ \ 2\ \ 1 \end{array}, \quad nqc(\mu) = \begin{array}{ccccc} c&q&q&q&0\\ c&q&q&0\\ c&q&0\\ c&0\\ 0 \end{array},$$

where, we have placed the Dynkin labels $a_1, ..., a_{m-1}$ *to the left of the first column, and in between the rows, of* $A(\mu)$, *likewise,* $a_{\dot 1}, ..., a_{\dot n-1}$ *are placed below the bottom row, and in between the columns, of* $A(\mu)$; *and where the numbered positions in* NE_μ *are located in the northeast chains* (*here and below we identify a matrix position* (a,b) *with the corresponding positive odd root* $\alpha_{a,\dot b}$). *The atypical roots are* $\gamma_1 = \alpha_{6,\dot 1} = \alpha_{\min}, \gamma_2 = \alpha_{5,\dot 2}, \gamma_3 = \alpha_{4,\dot 3}, \gamma_4 = \alpha_{2,\dot 4}, \gamma_5 = \alpha_{1,\dot 5} = \alpha_{\max}$, *and*

$$\Lambda_\mu = \mu + \sum_{\alpha \in NE_\mu} \alpha = (5,5,4,3,3,2 \,|\, -2,-3,-5,-6,-6).$$

Example A.5. $\mu = (5,2,2,1,1,1 \,|\, -1,-1,-1,-3,-6) = [3,0,1,0,0;0;0,0,2,3]$.

$$A(\mu) = \begin{array}{c} 3\\0\\1\\0\\0 \end{array} \begin{array}{c} \begin{pmatrix} 9&8&7&4&0\\ 5&4&3&0&\bar4\\ 4&3&2&\bar1&\bar5\\ 2&1&0&\bar3&\bar7\\ 1&0&\bar1&\bar4&\bar8\\ 0&\bar1&\bar2&\bar5&\bar9 \end{pmatrix} \\ 0\ \ 0\ \ 2\ \ 3 \end{array}, \quad NE_\mu = \begin{array}{c} 3\\0\\1\\0\\0 \end{array} \begin{array}{c} \begin{pmatrix} *&*&*&*&5\\ *&1&2&4&*\\ *&1&2&2&*\\ 1&2&3&1&*\\ 1&2&2&*&*\\ 1&1&1&*&* \end{pmatrix} \\ 0\ \ 0\ \ 2\ \ 3 \end{array}, \quad nqc(\mu) = \begin{array}{ccccc} q&n&n&n&0\\ c&c&q&0\\ c&c&0\\ c&0\\ 0 \end{array}.$$

The atypical roots are $\gamma_1 = \alpha_{\min}, \gamma_2 = \alpha_{5,\dot 2}, \gamma_3 = \alpha_{4,\dot 3}, \gamma_4 = \alpha_{2,\dot 4}, \gamma_5 = \alpha_{1,\dot 5} = \alpha_{\max}$, *and*

$$\Lambda_\mu = \mu + \sum_{\alpha \in NE_\mu} \alpha = (6,5,5,5,4,4 \,|\, -4,-6,-6,-6,-7).$$

For a weight μ of the form (5.5), its atypical roots are

$$\gamma_1 = \alpha_{\min} = \alpha_{m,\dot 1}, ..., \gamma_{n-k} = \alpha_{m-n+k+1, \dot n-k}, \quad \gamma_{n-k+1} = \alpha_{k, \dot n-k+1}, ..., \gamma_n = \alpha_{1,\dot n} = \alpha_{\max}.$$

We denote

$$P_\mu = \{\alpha_{i,\dot\nu} \in \Delta_{\bar 1}^+ \,|\, \nu \geq \mu_i - \mu_m \text{ and } i \leq \mu_{\dot 1} - \mu_{\dot\nu}\}. \tag{A.5}$$

Note that

$$\begin{aligned} \#P_\mu &= mn - \Big(\sum_{i=1}^m (\mu_i - \mu_m) + \sum_{\nu=1}^n (\mu_{\dot 1} - \mu_{\dot\nu})\Big) \\ &= mn - 2\,top + m\mu_m - n\mu_{\dot 1} = mn - 2\,top + (m+n)\mu_m. \end{aligned} \tag{A.6}$$

From the definition of NE_μ, one can easily obtain

Lemma A.6. (1) $P_\mu \subset NE_\mu$; (2) $P_\mu = NE_\mu \iff \gamma_1$ *is c- or q-related to any other atypical roots.* □

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